



Characterization of coorbit spaces with phase-space covers

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Abstract

We show that coorbit spaces can be characterized in terms of arbitrary phase-space covers, which are families of phase-space multipliers associated with partitions of unity. This generalizes previously known results for time-frequency analysis to include time-scale decompositions. As a by-product, we extend the existing results for time-frequency analysis to an irregular setting.

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1. Introduction

Coorbit spaces are functional spaces defined by imposing size conditions to a certain transform. More precisely, regarding a functional space X as a coorbit space consist of giving a transform $T : X \rightarrow E$ that embeds X into another functional space E that is *solid*. This means that the membership in E is determined by size conditions (for precise definitions see Section 2.2). The

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space E consists of functions defined on a set \mathcal{G} that is commonly taken to be a locally compact group.

The theory in [15] studies the case when T arises as the representation coefficients of a unitary action of a locally compact group. The examples of this theory include a wide range of classical function spaces. In the case of the affine group acting on $L^2(\mathbb{R}^d)$ by translations and dilations, T is the so-called continuous wavelet transform and the corresponding class of coorbit spaces includes the Lebesgue spaces L^p ($1 < p < \infty$), Sobolev spaces and, more generally, the whole class of Besov and Triebel–Lizorkin spaces. In the case of the Heisenberg group acting on $L^2(\mathbb{R}^d)$ by time-frequency shifts, the transform T is known as the short-time Fourier transform (or windowed Fourier transform) and the corresponding coorbit spaces are known as modulation spaces [13,17].

When a functional space X is identified as a coorbit space, the properties of an element $f \in X$ are reformulated in terms of decay or integrability conditions of the function $T(f) \in E$, that is sometimes referred to as the *phase-space representation* of f . The elements of X can be resynthesized from their phase-space representations by means of an operator $W : E \rightarrow X$ that is a left-inverse for T (i.e. $f = WT(f)$). In an attempt to finely adjust the properties of a function f that are expressed by $T(f)$ one can consider operators of the form $M_m(f) = W(mT(f))$ that apply a mask m to the phase-space representation $T(f)$. We will call these operators *phase-space multipliers*. Of course, the rigorous interpretation of $M_m(f)$ is problematic since, in general, $TM_m(f) \neq mT(f)$. When T is the abstract wavelet transform (representation-coefficients function) associated with a unitary action of a group, these operators are known as *localization operators* or *wavelet multipliers* [36,50,39]. In the case of time-frequency analysis these operators are known as time-frequency localization operators or multipliers of the short-time Fourier transform [8,5,6,3].

In this article we characterize the norm of a coorbit space in terms of families of phase-space multipliers associated with an arbitrary partition of unity in \mathcal{G} . Specifically, suppose that X is a Banach space that is regarded as a coorbit space by means of a transform $T : X \rightarrow E$, having a left-inverse $W : E \rightarrow X$. Let $\{\theta_\gamma\}_\gamma$ be a partition of unity on \mathcal{G} and consider the corresponding phase-space multipliers given by $M_\gamma(f) = W(\theta_\gamma T(f))$. The partition of unity is only assumed to satisfy certain spatial localization conditions but it is otherwise arbitrary. We prove that $\|f\|_X$ is equivalent to the norm of the sequence $\{\|M_\gamma(f)\|_B\}_\gamma$ in a discrete version of the space E , where the space B can be chosen among a large class of function spaces. Moreover, we prove that the map $f \mapsto \{M_\gamma(f)\}_\gamma$ embeds X as a complemented subspace of a space of B -valued sequences, obtained as a discretization of E . (See Theorem 3 for a precise statement.) This quantifies the relation between an element $f \in X$ and the phase-space localized pieces $\{M_\gamma(f)\}_\gamma$.

Phase-space multipliers formalize the notion of acting on a vector by operating on its phase-space representation. The set of all phase-space representations $T(X)$ is usually thought of as the class of all functions on phase-space, and phase-space itself is implicitly understood as the underlying set of “degrees of freedom” for that class of functions. Operations on phase-space are formally described as operations on the class of functions $T(X)$. Thus, a family of phase-space multipliers $\{M_\gamma\}_\gamma$ associated with a partition of unity $\{\theta_\gamma\}_\gamma$ on \mathcal{G} is a natural formalization of the notion of a cover of phase-space. From this perspective, the estimates we prove, that establish a quantitative equivalence between a vector f and the sequence $\{M_\gamma(f)\}_\gamma$, can be interpreted as saying that the family of operators $\{M_\gamma\}_\gamma$ indeed covers phase-space.

For the case of time-frequency analysis, Dörfler, Feichtinger and Gröchenig [9,10] have recently obtained a characterization of modulation spaces through families of time-frequency localization operators, using techniques from rotation algebras (non-commutative tori) devel-

oped in [31] and [30] and spectral theory for Hilbert spaces.² In this article we use a different approach to obtain consequences for settings where the techniques in [10] are not applicable, such as time-scale decompositions and Besov spaces. As a by-product we derive a stronger version of the main result in [10] where the admissible partitions of unity are restricted to be lattice shifts of a non-negative function and the space \mathbf{B} is L^2 . (For precise statements see Section 9.3.)

We now comment on the organization of the article. We consider an abstract setting in which there is a solid space \mathbf{E} of functions over a group \mathcal{G} and a certain complemented subspace \mathbf{S} (this is similar to the setting studied in [41]). Phase-space multipliers are defined as operators of the form $\mathbf{S} \ni f \mapsto P(mf) \in \mathbf{S}$ where $P : \mathbf{E} \rightarrow \mathbf{S}$ is a projection and $m \in L^\infty(\mathcal{G})$. The main result we prove is the characterization of the norm of \mathbf{S} in terms of the family of multipliers associated with an arbitrary partition of unity in \mathcal{G} (see Theorem 3). The technique of the proof is a vector-valued variant of the proof of the existence of atomic decompositions for coorbit spaces in [15]. In Section 9 we apply the abstract results to coorbit spaces, by taking \mathbf{S} to be the range of an adequate transform. The model for phase-space that we consider includes the classical coorbit theory associated with a group representation but also the case of coorbit spaces produced from localized frames (see [21]). This yields further applications to time-frequency analysis, giving a characterization of modulation spaces in terms of certain discrete time-frequency localization operators known as Gabor multipliers (see [18]).

Amalgam spaces (see Section 2.3) are one of the main technical tools of this article. We use and slightly extend (see Section 2.4) a number of convolution and sampling relations from [15] that are particularly important to discretization of convolution operators.

Section 8 establishes a variant of the main result where, under stronger assumptions on the group \mathcal{G} , the class of admissible partitions of unity is enlarged. This partial extension of the main result is important in a number of examples and, in particular, allows us to recover and extend the main result from [10]. Instead of the tools from rotation algebras used there, we resort to related results for matrix algebras.

2. Preliminaries

2.1. Notation

Throughout the article \mathcal{G} will be a locally compact, σ -compact, topological group with identity element e and modular function Δ . The left Haar measure of a set $X \subseteq \mathcal{G}$ will be denoted by $|X|$ whereas its cardinality will be denoted by $\#X$. Integration will be always considered with respect to the left Haar measure. The product of two subsets of \mathcal{G} , A , B , will be denoted by $A \cdot B$ or simply AB .

For $x \in \mathcal{G}$, we denote by L_x and R_x the operators of left and right translation, defined by $L_x f(y) = f(x^{-1}y)$ and $R_x f(y) = f(yx)$. We also consider the involution $f^\vee(x) = f(x^{-1})$.

Given two non-negative functions f, g we write $f \lesssim g$ if there exists a constant $C \geq 0$ such that $f \leq Cg$. We say that $f \approx g$ if both $f \lesssim g$ and $g \lesssim f$. The characteristic function of the set A will be denoted by χ_A . The symbol $\langle \cdot, \cdot \rangle$ will stand for the L^2 inner product, $\langle f, g \rangle := \int_{\mathcal{G}} f(x)\overline{g(x)} dx$, whenever defined.

² For more about the relation between time-frequency analysis and non-commutative tori see [40].

A set $\Lambda \subseteq \mathcal{G}$ is called *relatively separated* if for some (or any) $V \subseteq \mathcal{G}$, relatively compact neighborhood of e , the quantity – called the *spreadness* of Λ –

$$\rho(\Lambda) = \rho_V(\Lambda) := \sup_{x \in \mathcal{G}} \#(\Lambda \cap xV) \quad (1)$$

is finite, i.e. if the amount of elements of Λ that lie in any left translate of V is uniformly bounded. Equivalently, Λ is relatively separated if for any compact set $K \subseteq \mathcal{G}$,

$$\sup_{\lambda \in \Lambda} \# \{ \lambda' \in \Lambda \mid \lambda K \cap \lambda' K \neq \emptyset \} < +\infty.$$

A set $\Lambda \subseteq \mathcal{G}$ is called *V-dense* (for V , a relatively compact neighborhood of e) if $\mathcal{G} = \bigcup_{\lambda \in \Lambda} \lambda V$. Λ is called *well-spread* if it is both relatively separated and *V-dense* for some V .

We now fix V , a symmetric (i.e. $V = V^{-1}$) relatively compact neighborhood of the identity in \mathcal{G} . Some definitions below depend on the choice of V , but different choices of V will yield equivalent objects.

We will sometimes assume that \mathcal{G} is an IN group, i.e., that it has a relatively compact neighborhood of the identity that is invariant under inner automorphisms. By convention, whenever we assume that \mathcal{G} is an IN group we will further assume that the distinguished neighborhood V is invariant (i.e. $xVx^{-1} = V$, for all $x \in \mathcal{G}$).

2.2. Function spaces

A *BF space* is a Banach space E consisting of functions on \mathcal{G} that is continuously embedded into $L^1_{\text{loc}}(\mathcal{G})$, the space of locally integrable functions.

A BF space E is called *solid* if for every $f \in E$ and every measurable function $g: \mathcal{G} \rightarrow \mathbb{C}$ such that $|g(x)| \leq |f(x)|$ a.e., it is true that $g \in E$ and $\|g\|_E \leq \|f\|_E$.

An *admissible weight* is a locally bounded function $w: \mathcal{G} \rightarrow (0, +\infty)$ that satisfies the following conditions,

$$w(x) = \Delta(x^{-1})w(x^{-1}), \quad (2)$$

$$w(xy) \leq w(x)w(y) \quad (\text{submultiplicativity}). \quad (3)$$

If E is a solid BF space and w is an admissible weight, we let E_w be the set of all functions $f \in L^1_{\text{loc}}(\mathcal{G})$ such that $fw \in E$ and endow it with the norm $\|f\|_{E_w} := \|fw\|_E$. If w is an admissible weight, then L^1_w is a convolution algebra, $\|f\|_{L^1_w} = \|f^\vee\|_{L^1_w}$, and $\|L_x\|_{L^1_w \rightarrow L^1_w} \leq w(x)$.

We say that a solid BF space E is *translation invariant* if it satisfies the following.

- (i) E is closed under left and right translations (i.e. $L_x E \subseteq E$ and $R_x E \subseteq E$, for all $x \in \mathcal{G}$).
- (ii) The relations,

$$L^1_u(\mathcal{G}) * E \subseteq E \quad \text{and} \quad E * L^1_v \subseteq E, \quad (4)$$

hold, with the corresponding norm estimates, where $u(x) := \|L_x\|_{E \rightarrow E}$ and $v(x) := \|\Delta(x^{-1})\|_{R_{x^{-1}}\|_{E \rightarrow E}}$.

Remark 1. Observe that, for a BF space, if the translations leave \mathbf{E} invariant, then they are bounded by the closed graph theorem.

Remark 2. In the definition of translation invariant space, the technical assumption (ii) follows from (i) if the set of continuous functions with compact support is dense on \mathbf{E} , or more generally if the maps $x \mapsto L_x$ and $x \mapsto R_x$ are strongly continuous.

We say that \mathbf{E} is isometrically left (right) translation invariant if it is translation invariant and, in addition, left (right) translations are isometries on \mathbf{E} . The weighted Lebesgue spaces $L_m^p(\mathbb{R}^d)$ with $m(x) := (1 + |x|)^\alpha$, $\alpha \in \mathbb{R}$, are examples of translation invariant solid BF spaces on \mathbb{R}^d . These are isometrically translation invariant if $m \equiv 1$.

Given a BF space \mathbf{E} , a set of functions $\{f_\lambda \mid \lambda \in \Lambda\} \subseteq L_{\text{loc}}^1(\mathcal{G})$ – indexed by a relatively separated set Λ – is called a set of \mathbf{E} -molecules if there exists a function $g \in \mathbf{E}$ – called envelope – such that

$$|f_\lambda(x)| \leq L_\lambda g(x) \quad (x \in \mathcal{G}, \lambda \in \Lambda).$$

Given a solid, translation invariant, BF space \mathbf{E} , we say that a weight $w: \mathcal{G} \rightarrow (0, +\infty)$ is *admissible for \mathbf{E}* if w is admissible and, in addition, it satisfies

$$w(x) \geq C_{E,w} \max\{u(x), u(x^{-1}), v(x), \Delta(x^{-1})v(x^{-1})\}, \quad (5)$$

where $u(x) := \|L_x\|_{\mathbf{E} \rightarrow \mathbf{E}}$, $v(x) := \|\Delta(x^{-1})\|_{R_{x^{-1}}\|_{\mathbf{E} \rightarrow \mathbf{E}}$ and $C_{E,w} > 0$ is a constant. Under these conditions, $w(x) \gtrsim 1$, $L_w^1 * \mathbf{E} \subseteq \mathbf{E}$ and $\mathbf{E} * L_w^1 \subseteq \mathbf{E}$, with the corresponding norm estimates. Moreover, the constants in those estimates depend only on $C_{E,w}$.

If \mathbf{E} is a solid BF space, we construct discrete versions of it as follows. Given a well-spread set $\Lambda \subseteq \mathcal{G}$ we define the space,

$$\mathbf{E}_d = \mathbf{E}_d(\Lambda) := \left\{ c \in \mathbb{C}^\Lambda \mid \sum_\lambda |c_\lambda| \chi_{\lambda V} \in \mathbf{E} \right\},$$

and endow it with the norm,

$$\|(c_\lambda)_{\lambda \in \Lambda}\|_{\mathbf{E}_d} := \left\| \sum_\lambda |c_\lambda| \chi_{\lambda V} \right\|_{\mathbf{E}}.$$

The definition, of course, depends on Λ and V , but a large class of neighborhoods V and sets Λ produce equivalent spaces (see [15, Lemma 3.5] for a precise statement). In the sequel, we will mainly use the space \mathbf{E}_d keeping V fixed and making an explicit choice for Λ . When $\mathbf{E} = L_w^p$, for an admissible weight w , the corresponding discrete space $\mathbf{E}_d(\Lambda)$ is $\ell_w^p(\Lambda)$, where the weight w is restricted to the set Λ . This is so because the admissibility of w implies that for $x \in \lambda V$, $w(x) \approx w(\lambda)$.

We will also need a vector-valued version of \mathbf{E}_d . Given a second Banach space \mathbf{B} we let,

$$\begin{aligned}\mathbf{E}_{d,\mathbf{B}} = \mathbf{E}_{d,\mathbf{B}}(\Lambda) &:= \{c \in \mathbf{B}^\Lambda \mid (\|c_\lambda\|_{\mathbf{B}})_{\lambda \in \Lambda} \in \mathbf{E}_d(\Lambda)\} \\ &= \left\{c \in \mathbf{B}^\Lambda \mid \sum_{\lambda} \|c_\lambda\|_{\mathbf{B}} \chi_{\lambda V} \in \mathbf{E}\right\},\end{aligned}$$

and endow it with a norm in a similar fashion.

2.3. Wiener amalgam spaces

See [11,37,22]. Given two solid, translation invariant BF spaces \mathbf{B} and \mathbf{E} , the left *amalgam space* (or space of Wiener-type) with local component \mathbf{B} and global component \mathbf{E} is defined by

$$W(\mathbf{B}, \mathbf{E}) := \{f \in \mathbf{B}_{\text{loc}} \mid K_{\mathbf{B}}(f) \in \mathbf{E}\},$$

where the control function $K_{\mathbf{B}}(f)$ is given by $K_{\mathbf{B}}(f)(x) := \|f(L_x \chi_V)\|_{\mathbf{B}} = \|f \chi_{xV}\|_{\mathbf{B}}$. We endow $W(\mathbf{B}, \mathbf{E})$ with the norm $\|f\|_{W(\mathbf{B}, \mathbf{E})} := \|K_{\mathbf{B}}(f)\|_{\mathbf{E}}$. The right amalgam space $W_R(\mathbf{B}, \mathbf{E})$ is defined similarly, this time using the control function $K_{\mathbf{B}}(f, R)(x) := \|f(R_x \chi_V)\|_{\mathbf{B}} = \|f \chi_{Vx^{-1}}\|_{\mathbf{B}}$. This definition depends on the choice of the neighborhood V , but a different choice produces the same space with an equivalent norm.

When \mathbf{B} and \mathbf{E} are weighted L^p spaces, the corresponding amalgam space coincides with the classical $L^p - \ell^q$ amalgam spaces [37,22]. We are requiring that the space \mathbf{B} be solid, but much of the theory only requires that \mathbf{B} have a sufficiently rich algebra of pointwise multipliers (see [11]).

Amalgam spaces with L^1 and L^∞ as local components will be a key technical tool in this article. The spaces $W(L^\infty, \mathbf{E})$ and $W_R(L^\infty, \mathbf{E})$ can be easily described in terms of certain maximum functions. For a locally bounded function $f: \mathcal{G} \rightarrow \mathbb{C}$ we define the left and right *local maximum* functions by

$$f^\#(x) := \sup_{y \in V} |f(xy)|,$$

$$f_\#(x) := \sup_{y \in V} |f(yx)|.$$

Since V is symmetric, these functions are related by $(f_\#)^\vee = (f^\#)^\#$. Using these definitions we have

$$\|f\|_{W(L^\infty, \mathbf{E})} = \|f^\#\|_{\mathbf{E}}, \quad (6)$$

$$\|f\|_{W_R(L^\infty, \mathbf{E})} = \|(f_\#)^\vee\|_{\mathbf{E}}. \quad (7)$$

In particular, $\|f\|_{W_R(L^\infty, \mathbf{E})} = \|f^\vee\|_{W(L^\infty, \mathbf{E})}$. Note also that, by the solidity of \mathbf{E} , both $W(L^\infty, \mathbf{E})$ and $W_R(L^\infty, \mathbf{E})$ are continuously embedded into \mathbf{E} . We will denote by $W(C_0, \mathbf{E})$ the subspace of $W(L^\infty, \mathbf{E})$ formed by the continuous functions. $W_R(C_0, \mathbf{E})$ is defined similarly.

When $\mathbf{E} = L^1_w$ for an admissible weight w we can drop the involution in Eq. (7) yielding

$$\|f\|_{W_R(L^\infty, L^1_w)} := \|f_\#\|_{L^1_w}. \quad (8)$$

In addition, since $L_x(f^\#) = (L_x f)^\#$, the space $W(L^\infty, L_w^1)$ is invariant under left translations and the norm of the left translations is dominated by w . A similar statement holds for $W_R(L^\infty, L_w^1)$ and right translations.

We finally note that if \mathcal{G} is an IN group, the left and right local maximum functions coincide and therefore $W(L^\infty, E) = W_R(L^\infty, E)$.

We now state some facts about amalgam spaces that are relevant to atomic decompositions. They have been mainly collected from [15]. In the cases when we were unable to find an exact reference we sketch a proof. Most of the results below concern a translation invariant BF space E and an admissible weight w . We point out that the corresponding estimates depend only on w and certain qualities of E , namely the value of the constant $C_{E,w}$ in Eq. (5).

Lemma 1. *Let E be a solid, translation invariant BF space and let w be an admissible weight for it. The following embeddings hold, together with the corresponding norm estimates.³*

- (a) $E * W(L^\infty, L_w^1) \hookrightarrow W(L^\infty, E)$ and $E * W(C_0, L_w^1) \hookrightarrow W(C_0, E)$.
- (b) $E \hookrightarrow W(L^1, L_{1/w}^\infty)$. In addition, if E is isometrically left translation invariant, then $E \hookrightarrow W(L^1, L^\infty)$.
- (c) $W(L^1, L^\infty) \cdot W(L^\infty, L^1) \hookrightarrow L^1$ and $W(L^1, L_{1/w}^\infty) \cdot W(L^\infty, L_w^1) \hookrightarrow L^1$.
- (d) $W(L^1, L^\infty) * W_R(L^\infty, L^1) \hookrightarrow L^\infty$ and $W(L^1, L_{1/w}^\infty) * W_R(L^\infty, L_w^1) \hookrightarrow L_{1/w}^\infty$.

Proof. Part (a) is proved in [16, Theorem 7.1]. By [15, Lemma 3.9], $E \hookrightarrow W(L^1, L_{(u^\vee)^{-1}}^\infty)$, where $u(x) := \|L_x\|_{E \rightarrow E}$. The admissibility of w implies that $u^\vee \lesssim w$, so part (b) follows.

To prove (c) first observe that for any $f \in L^1(\mathcal{G})$, since $V = V^{-1}$,

$$\int_{\mathcal{G}} \int_{\mathcal{G}} |f(x)| (L_y \chi_V(x)) dx dy = \int_{\mathcal{G}} |f(x)| \int_{\mathcal{G}} \chi_V(y^{-1}x) dy dx = |V| \int_{\mathcal{G}} |f(x)| dx.$$

Using this observation, for $f \in W(L^1, L_{1/w}^\infty)$ and $g \in W(L^\infty, L_w^1)$,

$$\begin{aligned} \int_{\mathcal{G}} |f(x)| |g(x)| dx &\approx \int_{\mathcal{G}} \int_{yV} |f(x)| |g(x)| dx dy \\ &\lesssim \int_{\mathcal{G}} \|f\|_{L^1(yV)} \|g\|_{L^\infty(yV)} dy \leq \|f\|_{W(L^1, L_{1/w}^\infty)} \|g\|_{W(L^\infty, L_w^1)}. \end{aligned}$$

The unweighted case follows similarly. To prove (d) let $f \in W(L^1, L_{1/w}^\infty)$ and $g \in W_R(L^\infty, L_w^1)$. For $x \in \mathcal{G}$ we can use (c) to get

$$\begin{aligned} |f * g(x)| &\leq \int_{\mathcal{G}} |f(y)| |L_x g^\vee(y)| dy \leq \|f\|_{W(L^1, L_{1/w}^\infty)} \|L_x g^\vee\|_{W(L^\infty, L_w^1)} \\ &\leq \|f\|_{W(L^1, L_{1/w}^\infty)} \|g^\vee\|_{W(L^\infty, L_w^1)} w(x). \end{aligned}$$

³ Here, the symbol \cdot denotes the pointwise product.

Since $\|g^\vee\|_{W(L^\infty, L^1_w)} = \|g\|_{W_R(L^\infty, L^1_w)}$ the weighted inequality in (d) is proved. The unweighted one follows similarly, this time using the unweighted bound in (c). \square

Lemma 2. *Let E be a solid, translation invariant BF space, let w be an admissible weight for it and let $\Lambda \subseteq \mathcal{G}$ be a relatively separated set. Then the following hold.*

(a) *For every $f \in W(C_0, E)$, the sequence $f(\Lambda) = (f(\lambda))_{\lambda \in \Lambda}$ belongs to $E_d(\Lambda)$ and*

$$\|f(\Lambda)\|_{E_d} \lesssim \|f\|_{W(C_0, E)}.$$

(b) *For every $f \in E$ and $g \in W_R(C_0, L^1_w)$, the sequence $(\langle f, L_\lambda g \rangle)_{\lambda \in \Lambda}$ belongs to $E_d(\Lambda)$ and*

$$\|(\langle f, L_\lambda g \rangle)_\lambda\|_{E_d} \lesssim \|f\|_E \|g\|_{W_R(L^\infty, L^1_w)}.$$

(c) *If $(c_\lambda)_\lambda \in E_d(\Lambda)$ and $f \in W_R(L^\infty, L^1_w)$, then $\sum_\lambda c_\lambda L_\lambda f \in E$ and*

$$\left\| \sum_\lambda c_\lambda L_\lambda f \right\|_E \lesssim \|(c_\lambda)_\lambda\|_{E_d} \|f\|_{W_R(L^\infty, L^1_w)}.$$

The series converges absolutely at every point and, if the set of bounded compactly supported functions is dense in E , it also converges unconditionally in the norm of E .

(d) $E_d(\Lambda) \hookrightarrow \ell_{1/w}^\infty(\Lambda)$.

All the implicit constants depend on $\rho(\Lambda)$ – the spreadness of Λ (cf. Eq. (1)).

Proof. Part (a) follows easily from the definitions (see for example [15, Lemma 3.8]). For (b) observe that $\langle f, L_\lambda g \rangle = (f * g^\vee)(\lambda)$. Hence, Lemma 1 and part (a) imply that

$$\|(\langle f, L_\lambda g \rangle)_\lambda\|_{E_d} \lesssim \|f * g^\vee\|_{W(C_0, E)} \lesssim \|f\|_E \|g^\vee\|_{W(L^\infty, L^1_w)} = \|f\|_E \|g\|_{W_R(L^\infty, L^1_w)}.$$

Part (c) is Proposition 5.2 of [15]. Lemma 3.5 in [15] gives the embedding $E_d(\Lambda) \hookrightarrow \ell_{1/u}^\infty(\Lambda)$, where $u(x) := \|L_x\|_{E \rightarrow E}$. Since $u \lesssim w$, part (d) follows. \square

Finally we state the following lemma that will be used to justify treating convolutions point-wise.

Lemma 3. *Let E be a solid, translation invariant BF space and let w be an admissible weight for it. The following embeddings hold, together with the corresponding norm estimates.*

(a) $W(L^\infty, E) \hookrightarrow L_{1/w}^\infty$.

(b) $W(L^\infty, E) * L_w^1 \hookrightarrow C_{1/w}$, where $C_{1/w}$ denotes the subspace of $L_{1/w}^\infty$ formed by the continuous functions.

Proof. By Lemma 2, $E_d \hookrightarrow \ell_{1/w}^\infty$. This implies that,

$$W(L^\infty, E) \hookrightarrow W(L^\infty, L_{1/w}^\infty) = L_{1/w}^\infty$$

(see for example [15, Proposition 3.7]). This proves part (a). The embedding,

$$W(L^\infty, E) \cdot L_w^1 \hookrightarrow L_{1/w}^\infty \cdot L_w^1 \hookrightarrow L^1,$$

implies that, $W(L^\infty, E) * L_w^1 = W(L^\infty, E) * L_w^{1\vee} \hookrightarrow L_{1/w}^\infty$. Now part (b) follows from the fact that the class of continuous, compactly supported functions is dense in L_w^1 . \square

2.4. Weak and strong amalgam norms

We now introduce some variations of the amalgam spaces $W(L^\infty, L_w^1)$, $W_R(L^\infty, L_w^1)$. We do so in order to deal with some technicalities involving right convolution actions on the spaces $W(L^\infty, E)$. For an IN group, the spaces $W(L^\infty, E)$ are right L_w^1 modules, but for a general group \mathcal{G} , they are only right $W(L^\infty, L_w^1)$ modules. We will now introduce a space between L_w^1 and $W(L^\infty, L_w^1)$ that acts on the spaces $W(L^\infty, E)$ from the right and collapses to L_w^1 in the case that \mathcal{G} is an IN group.

Similarly, we will define a certain subspace of $W(L^\infty, L_w^1) \cap W_R(L^\infty, L_w^1)$ that reduces to $W(L^\infty, L_w^1)$ when \mathcal{G} is an IN group. The introduction of this second space is not essential but a matter of convenience. Its use is not required by any of the applications in Section 9.

For an admissible weight w , let the left and right *weak amalgam spaces* be defined by

$$\begin{aligned} W^{\text{weak}}(L^\infty, L_w^1) &:= \{f \in L_{\text{loc}}^1 \mid \chi_V * |f| \in W(L^\infty, L_w^1)\}, \\ W_R^{\text{weak}}(L^\infty, L_w^1) &:= \{f \in L_{\text{loc}}^1 \mid |f| * \chi_V \in W_R(L^\infty, L_w^1)\}, \end{aligned}$$

and endow them with the norms,

$$\begin{aligned} \|f\|_{W^{\text{weak}}(L^\infty, L_w^1)} &:= \|\chi_V * |f|\|_{W(L^\infty, L_w^1)} = \|(\chi_V * |f|)^\# \|_{L_w^1}, \\ \|f\|_{W_R^{\text{weak}}(L^\infty, L_w^1)} &:= \||f| * \chi_V \|_{W_R(L^\infty, L_w^1)} = \|(|f| * \chi_V)_\# \|_{L_w^1}. \end{aligned}$$

These spaces are related by $\|f\|_{W^{\text{weak}}(L^\infty, L_w^1)} = \|f^\vee\|_{W_R^{\text{weak}}(L^\infty, L_w^1)}$.

Consider also the *strong amalgam space* defined as

$$W^{\text{st}}(L^\infty, L_w^1) := W_R(L^\infty, W(L^\infty, L_w^1)).$$

Hence, the norm of a function $f \in W^{\text{st}}(L^\infty, L_w^1)$ is given by

$$\|f\|_{W^{\text{st}}(L^\infty, L_w^1)} = \|(f^\#)^\# \|_{L_w^1}.$$

We now observe how these new spaces are related to the classical ones.

Proposition 1. *Let w be an admissible weight. Then the following hold.*

(a)

$$W(L^\infty, L_w^1) \hookrightarrow W^{\text{weak}}(L^\infty, L_w^1) \hookrightarrow L_w^1,$$

and

$$W_R(L^\infty, L_w^1) \hookrightarrow W_R^{\text{weak}}(L^\infty, L_w^1) \hookrightarrow L_w^1.$$

(b) If \mathcal{G} is an IN group, then

$$W_R^{\text{weak}}(L^\infty, L_w^1) = W^{\text{weak}}(L^\infty, L_w^1) = L_w^1.$$

(c) $W^{\text{st}}(L^\infty, L_w^1) \hookrightarrow W(L^\infty, L_w^1) \cap W_R(L^\infty, L_w^1)$.

(d) If \mathcal{G} is an IN group, then

$$W(L^\infty, L_w^1) = W_R(L^\infty, L_w^1) = W^{\text{st}}(L^\infty, L_w^1).$$

Proof. For (a) and (b) we only prove the statements concerning the “right” spaces; the corresponding statements for “left” spaces follow by using the involution $^\vee$.

Let $f \in W_R(L^\infty, L_w^1)$. Since $(|f| * \chi_V)_\# \leq (f_\# * \chi_V)$, we have that,

$$\begin{aligned} \|f\|_{W_R^{\text{weak}}(L^\infty, L_w^1)} &= \|(|f| * \chi_V)_\#\|_{L_w^1} \leq \|f_\# * \chi_V\|_{L_w^1} \\ &\leq \|f_\#\|_{L_w^1} \|\chi_V\|_{L_w^1} \lesssim \|f\|_{W_R(L^\infty, L_w^1)}. \end{aligned}$$

This proves the first embedding of (a). For the second one, let $f \in W_R^{\text{weak}}(L^\infty, L_w^1)$ and estimate

$$\begin{aligned} \int_{\mathcal{G}} |f(x)| w(x) dx &\lesssim \int_{\mathcal{G}} |f(x)| w(x) \int_{\mathcal{G}} \chi_V(x^{-1}y) dy dx \\ &\leq \int_{\mathcal{G}} \int_{\mathcal{G}} |f(x)| w(y^{-1}x) \chi_V(x^{-1}y) dx w(y) dy. \end{aligned}$$

Since w is locally bounded, in the last integral $w(y^{-1}x) \lesssim 1$ and we conclude that $\|f\|_{L_w^1} \lesssim \| |f| * \chi_V \|_{L_w^1}$. Now the conclusion follows from the fact that $|f| * \chi_V \leq (|f| * \chi_V)_\#$.

Part (b) follows from the convolution relation,

$$W(L^\infty, L_w^1) * L_w^1 \hookrightarrow W(L^\infty, L_w^1),$$

which holds when \mathcal{G} is an IN group. (This follows easily from the fact that, for an IN group, $f^\# = f_\#$, see for example [11, Theorem 3].)⁴

Part (c) follows from the observation that $f_\# \leq (f_\#)^\#$ and $f^\# \leq (f^\#)_\#$. Finally if \mathcal{G} is an IN group, for $x \in \mathcal{G}$, $VxV = VVx$, and therefore,

$$(f_\#)^\#(x) = \sup_{v \in V} f_\#(xv) = \sup_{v \in V} \sup_{w \in V} |f(wxv)| = \sup_{y \in VV} |f(yx)| = (f^\#)_\#(x). \quad (9)$$

⁴ Theorem 3 in [11] implies that $W(L^\infty, L_w^1) * W(L^1, L_w^1) \hookrightarrow W(L^\infty, L_w^1)$. It is straightforward to see that $W(L^1, L_w^1) = L_w^1$.

Hence, the conclusion follows from the fact that a different choice for the neighborhood V induces an equivalent norm in $W(L^\infty, L_w^1)$. \square

For the weak norm, we now derive the following convolution relation (cf. Lemma 1). Again, we point out that the estimates depend only on the weight w and the constant $C_{E,w}$ in Eq. (5).

Proposition 2. *Let E be a solid, translation invariant BF space and let w be an admissible weight for it. Then,*

$$W(L^\infty, E) * W^{\text{weak}}(L^\infty, L_w^1) \hookrightarrow W(C_0, E),$$

together with the corresponding norm estimate.

Proof. Let $f \in W(L^\infty, E)$ and $g \in W^{\text{weak}}(L^\infty, L_w^1)$. For almost every $y \in \mathcal{G}$ and $t \in V$, $|f(y)| \leq f^\#(yt)$. Hence, for $x \in \mathcal{G}$,

$$\begin{aligned} |f| * |g|(x) &\leq \int_{\mathcal{G}} \int_{\mathcal{G}} f^\#(yt) \chi_V(t^{-1}) dt |g(y^{-1}x)| dy \\ &= \int_{\mathcal{G}} f^\#(t) \int_{\mathcal{G}} \chi_V(t^{-1}y) |g(y^{-1}x)| dy dt \\ &= \int_{\mathcal{G}} f^\#(t) (\chi_V * |g|)(t^{-1}x) dt = f^\# * (\chi_V * |g|)(x). \end{aligned}$$

Therefore Lemma 1 implies that,

$$\begin{aligned} \|f * g\|_{W(L^\infty, E)} &\leq \|f^\# * (\chi_V * |g|)\|_{W(L^\infty, E)} \\ &\lesssim \|f^\#\|_E \|\chi_V * |g|\|_{W(L^\infty, L_w^1)} = \|f\|_{W(L^\infty, E)} \|g\|_{W^{\text{weak}}(L^\infty, L_w^1)}. \end{aligned}$$

It only remains to note that $f * g$ is a continuous function. This follows from the embedding $W^{\text{weak}}(L^\infty, L_w^1) \hookrightarrow L_w^1$ in Proposition 1 and Lemma 3. \square

Using Proposition 2, we can derive a variant of Lemma 2(b) that only requires g to be in $W_R^{\text{weak}}(L^\infty, L_w^1)$.

Lemma 4. *Let E be a solid, translation invariant BF space and let w be an admissible weight for it. Let $\Lambda \subseteq \mathcal{G}$ be a relatively separated set. Then, for $f \in W(L^\infty, E)$ and $g \in W_R^{\text{weak}}(L^\infty, L_w^1)$, the sequence $(\langle f, L_\lambda g \rangle)_{\lambda \in \Lambda}$ belongs to $E_d(\Lambda)$ and satisfies*

$$\|(\langle f, L_\lambda g \rangle)_\lambda\|_{E_d} \lesssim \|f\|_{W(L^\infty, E)} \|g\|_{W_R^{\text{weak}}(L^\infty, L_w^1)},$$

where the implicit constant depends on $\rho(\Lambda)$ (cf. Eq. (1)).

Proof. As in the proof on Lemma 2(b), $\|(\langle f, L_\lambda g \rangle)_\lambda\|_{E^d} \lesssim \|f * g^\vee\|_{W(C_0, E)}$. Now we can invoke Proposition 2 and the fact that the involution $^\vee$ maps $W_R^{\text{weak}}(L^\infty, L_w^1)$ into $W^{\text{weak}}(L^\infty, L_w^1)$ to obtain the desired conclusion. \square

3. The model for phase-space

We now introduce a general setting where there is a solid BF space E (called the environment) and a certain distinguished subspace S that is the range of an idempotent integral operator P .⁵ This is the natural setting for the results of this article and seems to be the easiest scenario to check in a number of concrete examples (see Section 9). In Section 7 we will consider a more particular setting where the subspace S has a distinguished atomic decomposition. This will allow us to make fine adjustments to the general results, as required by certain applications (see Section 8).

We list a number of ingredients in the form of two assumptions: (A1) and (A2).

- (A1) – E is a solid, translation invariant BF space, called *the environment*.
 – w is an admissible weight for E .
 – S is a closed complemented subspace of E , called *the atomic subspace*.

The second assumption is that the retraction $E \rightarrow S$ is given by an operator that is dominated by right convolution with a kernel in $W(L^\infty, L_w^1) \cap W_R(L^\infty, L_w^1)$.

- (A2) We have an operator P and a function H satisfying the following.
 – $P : W(L^1, L_{1/w}^\infty) \rightarrow L_{1/w}^\infty$ is a (bounded) linear operator,
 – $P(E) = S$,
 – $P(f) = f$, for all $f \in S$,
 – $H \in W(L^\infty, L_w^1) \cap W_R(L^\infty, L_w^1)$,
 – for $f \in W(L^1, L_{1/w}^\infty)$,

$$|P(f)(x)| \leq \int_{\mathcal{G}} |f(y)| H(y^{-1}x) dy \quad (x \in \mathcal{G}). \quad (10)$$

We now observe some consequences of these assumptions.

Proposition 3. *Under Assumptions (A1) and (A2) the following hold.*

- (a) P boundedly maps E into $W(L^\infty, E)$.
 (b) $S \hookrightarrow W(L^\infty, E)$.
 (c) If $f \in W(L^1, L_{1/w}^\infty)$, then $\|P(f)\|_{L_{1/w}^\infty} \lesssim \|f\|_{W(L^1, L_{1/w}^\infty)} \|H\|_{W_R(L^\infty, L_w^1)}$.
 (d) If $f \in W(L^1, L^\infty)$, then $\|P(f)\|_{L^\infty} \lesssim \|f\|_{W(L^1, L^\infty)} \|H\|_{W_R(L^\infty, L_w^1)}$.

Remark 3. Since $w \gtrsim 1$, $L^\infty \hookrightarrow L_{1/w}^\infty$.

⁵ This is similar to the setting studied in [41].

Proof of Proposition 3. Parts (a), (c) and (d) follow from Eq. (10), Lemma 1 and the fact that $w \gtrsim 1$. For (b), observe that by part (a), P maps E into $W(L^\infty, E)$ and coincides with the identity operator on S .

Remark 4. The estimates in Proposition 3 hold uniformly for all the spaces E with the same weight w and the same constant $C_{E,w}$ (cf. Eq. (5)).

In the applications the same projection P will be used with different spaces E and corresponding subspaces S , providing a unified analysis of a whole class of functional spaces. This is why Remark 4 is relevant. \square

4. Approximation of phase-space projections

In this section we prove the main technical estimate of the article. Given the setting from Section 3 and a partition of unity $\sum_\gamma \eta_\gamma \equiv 1$, we will show that the phase-space projection $P(f)$ from Section 3 can be resynthesized from the phase-space localized pieces $\{P(f\eta_\gamma)\}_\gamma$. Note that $P(f)$ can be trivially recovered from $\{P(f\eta_\gamma)\}_\gamma$ by simply summing all these functions. We will prove that this reconstruction can also be achieved by placing the localized pieces on top of the (morally) corresponding regions of the phase-space. This controlled synthesis will then allow us to quantify the relation between $P(f)$ and $\{P(f\eta_\gamma)\}_\gamma$ and yield the main result on the characterization of the norm of S .

4.1. Setting

Let us first formally introduce all the required ingredients. Suppose that Assumptions (A1) and (A2) from Section 3 hold. We now state Assumption (B1) introducing the partition of unity covering phase-space and the norm used to measure it.

- (B1) – $\Gamma \subseteq \mathcal{G}$ is a relatively separated set.
- $\{\eta_\gamma \mid \gamma \in \Gamma\}$ is a set of $W_R^{\text{weak}}(L^\infty, L_w^1)$ -molecules enveloped by a function g . More precisely,
 - $|\eta_\gamma(x)| \leq g(\gamma^{-1}x)$ ($x \in \mathcal{G}$, $\gamma \in \Gamma$),
 - $g \in W_R^{\text{weak}}(L^\infty, L_w^1)$.
 - $\{\eta_\gamma\}_\gamma$ is a bounded partition of unity. That is,

$$\sum_\gamma \eta_\gamma \equiv 1, \quad \text{and} \quad \sum_\gamma |\eta_\gamma| \in L^\infty(\mathcal{G}).$$

- B is a solid, isometrically left translation invariant Banach space such that $W(L^\infty, L_w^1) \hookrightarrow B$.

Remark 5. By Lemma 1, $B \hookrightarrow W(L^1, L^\infty)$. In addition, by the definition of translation invariant space $L^1 * B \hookrightarrow B$.

Remark 6. Note that the conditions in (B1) allow for the usual bounded uniform partitions of unity [11,14,12] – where functions are supported on a family of compact sets having a bounded number of overlaps – but also for functions having non-compact support.

4.2. Vector-valued analysis and synthesis

Let us now describe the operators mapping a function f into its phase-space localized pieces, by means of the partition of unity $\{\eta_\gamma\}_\gamma$. Let the *analysis operator* $C^{\mathbf{B}}$ be formally defined by

$$C^{\mathbf{B}}(f) := (P(f\eta_\gamma))_{\gamma \in \Gamma}. \quad (11)$$

For each U , a relatively compact neighborhood of the identity in \mathcal{G} , we also formally define the *synthesis operator* $S_U^{\mathbf{B}}$, acting on a sequence of functions by

$$S_U^{\mathbf{B}}((f_\gamma)_{\gamma \in \Gamma}) := \sum_\gamma P(f_\gamma)\chi_{\gamma U}. \quad (12)$$

The operator $S_U^{\mathbf{B}}$ will be used as an approximate left-inverse of the vector-valued analysis operator $C^{\mathbf{B}}$. Let us now establish the mapping properties of these operators.

Proposition 4. *Under Assumptions (A1), (A2) and (B1) the following statements hold.*

- (a) *The analysis operator $C^{\mathbf{B}}$ maps $W(L^\infty, \mathbf{E})$ boundedly into $\mathbf{E}_{d, \mathbf{B}}(\Gamma)$. In particular (cf. Proposition 3) it maps \mathbf{S} boundedly into $\mathbf{E}_{d, \mathbf{B}}(\Gamma)$.*
- (b) *For every relatively compact neighborhood of the identity U , and every sequence $F \in \mathbf{E}_{d, \mathbf{B}}$, the series defining $S_U^{\mathbf{B}}(F)$ converge absolutely in the norm of \mathbf{B} at every point. Moreover, the synthesis operator $S_U^{\mathbf{B}}$ maps $\mathbf{E}_{d, \mathbf{B}}(\Gamma)$ boundedly into \mathbf{E} (with a bound that depends on U).*

Proof. To prove (a) let $f \in W(L^\infty, \mathbf{E})$. Since η_γ is bounded, $f\eta_\gamma \in W(L^\infty, \mathbf{E}) \subseteq \mathbf{E}$. By the pointwise bound for P (cf. Eq. (10)),

$$\begin{aligned} |P(f\eta_\gamma)(x)| &\leq \int_{\mathcal{G}} |f(y)|g(\gamma^{-1}y)H(y^{-1}x)dy \\ &= (|f|L_\gamma g) * H(x). \end{aligned}$$

Since \mathbf{B} is solid and $L^1 * \mathbf{B} \hookrightarrow \mathbf{B}$, we have

$$\|P(f\eta_\gamma)\|_{\mathbf{B}} \leq \|H\|_{\mathbf{B}} \int_{\mathcal{G}} |f(y)|g(\gamma^{-1}y)dy \lesssim \|H\|_{W(L^\infty, L_w^1)} \int_{\mathcal{G}} |f(y)|g(\gamma^{-1}y)dy.$$

Now the solidity of \mathbf{E} and Lemma 4 yield

$$\|C^{\mathbf{B}}(f)\|_{\mathbf{E}_{d, \mathbf{B}}} \lesssim \|f\|_{W(L^\infty, \mathbf{E})} \|g\|_{W_R^{\text{weak}}(L^\infty, L_w^1)}.$$

To prove (b) consider a family $F \equiv (f_\gamma)_\gamma \in \mathbf{E}_{d, \mathbf{B}}$. For each $\gamma \in \Gamma$, $f_\gamma \in \mathbf{B} \subseteq W(L^1, L^\infty)$, so by Proposition 3, $P(f_\gamma)$ is well defined and satisfies

$$|P(f_\gamma)(x)| \lesssim \|f_\gamma\|_{W(L^1, L^\infty)} \|H\|_{W_R(L^\infty, L_w^1)} \lesssim \|f_\gamma\|_{\mathbf{B}} \|H\|_{W_R(L^\infty, L_w^1)}.$$

Hence, for every $x \in \mathcal{G}$, $|S^B(F)(x)| \lesssim \sum_{\gamma} \|f_{\gamma}\|_B \chi_U(\gamma^{-1}x)$. Since U is relatively compact, $\chi_U \in W_R(L^\infty, L_w^1)$ and consequently Lemma 2 together with the solidity of E imply that $\|S_U^B(F)\|_E \lesssim \|\chi_U\|_{W_R(L^\infty, L_w^1)} \|F\|_{E_{d,B}}$. \square

4.3. Approximation of the projector

Now we can state the main result on the approximation of P . For every U , relatively compact neighborhood of the identity in \mathcal{G} , consider the approximate projector $P_U : W(L^\infty, E) \rightarrow E$ given by

$$P_U(f) := \sum_{\gamma \in \Gamma} P(f\eta_{\gamma})\chi_{\gamma U}. \quad (13)$$

Since $P_U = S_U^B \circ C^B$, P_U is well defined. We will prove that P_U approximates P in the following way.

Theorem 1. *Given $\varepsilon > 0$, there exists U_0 , a relatively compact neighborhood of e such that for all $U \supseteq U_0$,*

$$\|P(f) - P_U(f)\|_E \leq \varepsilon \|f\|_{W(L^\infty, E)} \quad (f \in W(L^\infty, E)).$$

Remark 7. The neighborhood U_0 can be chosen uniformly for any class of spaces E having the same weight w and the same constant $C_{E,w}$ (cf. Eq. (5)).

Concerning the ingredients introduced in Assumptions (A2) and (B1), the choice of U_0 only depends on $\|H\|_{W(L^\infty, L_w^1)}$, $\|H\|_{W_R(L^\infty, L_w^1)}$, $\|g\|_{W_R^{\text{weak}}(L^\infty, L_w^1)}$ and $\rho(\Gamma)$ (cf. Eq. (1)).

In order to prove Theorem 1 we introduce the following auxiliary function. For each U , let $G_U : \mathcal{G} \rightarrow [0, +\infty)$ be defined by

$$G_U(x) := \sup_{y \in \mathcal{G}} \sum_{\gamma \in \Gamma} (g * \chi_V)(\gamma^{-1}y) \chi_{\gamma(\mathcal{G} \setminus U)}(yx). \quad (14)$$

Observe that G_U is defined as a supremum of a family of sums. The estimates for P that we will derive in terms of G_U are different from the usual convolution estimates involving Wiener amalgam norms of g and will be crucial for the proof of Theorem 1. Before proving that theorem we establish some necessary estimates for the auxiliary function.

Lemma 5. *The function G_U satisfies $\|G_U\|_{L^\infty(\mathcal{G})} \lesssim 1$ (with a bound independent of U). Moreover, for every compact set $K \subseteq \mathcal{G}$,*

$$\|G_{U \cdot K}\|_{L^\infty(K)} \lesssim \int_{V \cdot (\mathcal{G} \setminus U)} (g * \chi_V)_\#(x) w(x) dx.$$

Proof. Let a compact set K and an element $x \in K$ be given. For $y \in \mathcal{G}$, if $yx \in \gamma(\mathcal{G} \setminus (UK))$, then $\gamma^{-1}yx \notin UK$, so $\gamma^{-1}y \notin U$.

Therefore,

$$\begin{aligned} \sum_{\gamma} (g * \chi_V)(\gamma^{-1}y) \chi_{\gamma(\mathcal{G} \setminus (UK))}(yx) &\leq \sum_{\gamma: \gamma^{-1}y \notin U} (g * \chi_V)(\gamma^{-1}y) \\ &\lesssim \sum_{\gamma: \gamma^{-1}y \notin U} \int_{\mathcal{G}} (g * \chi_V)_{\#}(t^{-1}\gamma^{-1}y) \chi_V(t) dt \\ &= \int_{\mathcal{G}} (g * \chi_V)_{\#}(t^{-1}) \sum_{\gamma: \gamma^{-1}y \notin U} \chi_V(\gamma^{-1}yt) dt. \end{aligned}$$

Since Γ is relatively separated, $\sum_{\gamma} \chi_V(\gamma^{-1}yt) = \sum_{\gamma} \chi_V(t^{-1}y^{-1}\gamma) \lesssim 1$. In addition, if $\gamma^{-1}yt \in V$ and $\gamma^{-1}y \notin U$ then $t = (\gamma^{-1}y)^{-1}\gamma^{-1}yt \in (\mathcal{G} \setminus U)^{-1}V$.

Hence,

$$G_{U \cdot K}(x) \lesssim \int_{(\mathcal{G} \setminus U)^{-1} \cdot V} (g * \chi_V)_{\#}(t^{-1}) dt = \int_{V \cdot (\mathcal{G} \setminus U)} (g * \chi_V)_{\#}(t) \Delta(t^{-1}) dt.$$

Since $\Delta(t^{-1}) \lesssim \Delta(t^{-1})w(t^{-1}) = w(t)$ the desired bound follows. Reexamining the computations above we see that, $\|G_U\|_{L^\infty(\mathcal{G})} \lesssim \int_{\mathcal{G}} (g * \chi_V)_{\#}(t)w(t) dt$. Since $g \in W_R^{\text{weak}}(L^\infty, L_w^1)$, the last integral is finite and we get the desired uniform bound. \square

Now we can prove Theorem 1.

Proof of Theorem 1. Let $f \in W(L^\infty, E)$ and let U be a relatively compact neighborhood of e . Since $\sum_{\gamma} \eta_{\gamma} \equiv 1$,

$$P(f) - P_U(f) = \sum_{\gamma} P(f\eta_{\gamma}) - \sum_{\gamma} P(f\eta_{\gamma})\chi_{\gamma U} = \sum_{\gamma} P(f\eta_{\gamma})\chi_{\gamma(\mathcal{G} \setminus U)}.$$

Consequently, by the pointwise bound for P (cf. Eq. (10)), for $x \in \mathcal{G}$,

$$|P(f)(x) - P_U(f)(x)| \leq \sum_{\gamma} \int_{\mathcal{G}} |f(y)| g(\gamma^{-1}y) H(y^{-1}x) \chi_{\gamma(\mathcal{G} \setminus U)}(x) dy.$$

Since $|f(y)| \lesssim \int f^{\#}(z) \chi_V(y^{-1}z) dz$, we have

$$|P(f)(x) - P_U(f)(x)| \lesssim \int_{\mathcal{G}} f^{\#}(z) \sum_{\gamma} \int_{\mathcal{G}} \chi_V(y^{-1}z) g(\gamma^{-1}y) H(y^{-1}x) \chi_{\gamma(\mathcal{G} \setminus U)}(x) dy dz.$$

Observe that if $y^{-1}z \in V$, then $y^{-1}x = y^{-1}zz^{-1}x \in V(z^{-1}x)$, and therefore $H(y^{-1}x) \leq H_{\#}(z^{-1}x)$. Hence,

$$\begin{aligned}
|P(f)(x) - P_U(f)(x)| &\lesssim \int_{\mathcal{G}} f^\#(z) H_\#(z^{-1}x) \sum_{\gamma} \int_{\mathcal{G}} g(\gamma^{-1}y) \chi_V(y^{-1}z) \chi_{\gamma(\mathcal{G} \setminus U)}(x) dy dz \\
&= \int_{\mathcal{G}} f^\#(z) H_\#(z^{-1}x) \sum_{\gamma} (g * \chi_V)(\gamma^{-1}z) \chi_{\gamma(\mathcal{G} \setminus U)}(x) dz \\
&\leq \int_{\mathcal{G}} f^\#(z) H_\#(z^{-1}x) G_U(z^{-1}x) dz \\
&= f^\# * (H_\# G_U)(x).
\end{aligned}$$

Consequently,

$$\|P(f) - P_U(f)\|_E \leq \|f^\#\|_E \|H_\# G_U\|_{L_w^1} = \|f\|_{W(L^\infty, E)} \|H_\# G_U\|_{L_w^1}.$$

Therefore, it suffices to show that $\|H_\# G_U\|_{L_w^1} \rightarrow 0$, as U grows to \mathcal{G} . For every compact set $K \subseteq \mathcal{G}$, Lemma 5 implies that

$$\begin{aligned}
\int_{\mathcal{G}} H_\#(z) G_U(z) w(z) dz &\leq \|G_U\|_{L^\infty(K)} \|H_\#\|_{L_w^1} + \|G_U\|_{L^\infty(\mathcal{G})} \int_{\mathcal{G} \setminus K} H_\#(z) w(z) dz \\
&\lesssim \|G_U\|_{L^\infty(K)} + \int_{\mathcal{G} \setminus K} H_\#(z) w(z) dz.
\end{aligned}$$

Given $\varepsilon > 0$, we choose a compact set containing the identity K such that the second term in the last inequality is less than ε . Since $g \in W_R^{\text{weak}}(L^\infty, L_w^1)$, we can also choose a compact set containing the identity $Q \subseteq \mathcal{G}$ such that

$$\int_{\mathcal{G} \setminus Q} (g * \chi_V)_\#(x) w(x) dx < \varepsilon.$$

Set $U_0 := VQK$. If $U \supseteq U_0$ is a relatively compact neighborhood of e , then, using Lemma 5,

$$\|G_U(z)\|_{L^\infty(K)} \leq \|G_{VQK}\|_{L^\infty(K)} \lesssim \int_{V(\mathcal{G} \setminus (VQ))} (g * \chi_V)_\#(x) w(x) dx.$$

Since $V = V^{-1}$, we have that $V(\mathcal{G} \setminus (VQ)) \subseteq (\mathcal{G} \setminus Q)$ and consequently $\|G_U(z)\|_{L^\infty(K)} \lesssim \varepsilon$. Hence, we have shown that for $U \supseteq U_0$, $\|H_\# G_U\|_{L_w^1} \lesssim \varepsilon$. This completes the proof. \square

5. Approximation of phase-space multipliers

We will now interpret Theorem 1 as a result about approximation of *phase-space multipliers*. Let us suppose that Assumptions (A1), (A2) and (B1) hold.

For $m \in L^\infty(\mathcal{G})$, the multiplier $M_m : \mathcal{S} \rightarrow \mathcal{S}$ with symbol m is defined by

$$M_m(f) := P(mf) \quad (f \in \mathcal{S}). \quad (15)$$

The operator M_m is clearly bounded by Proposition 3 and the solidity of \mathcal{E} . When the space \mathcal{S} is taken to be the range of the abstract wavelet transform associated with a unitary representation of \mathcal{G} , these operators are called localization operators or wavelet multipliers (see for example [36, 50, 39]). (More precisely, the operators M_m are unitary equivalent to localization operators, see Section 9.1 for further details). When \mathcal{S} is the range of the short-time Fourier transform the corresponding operators are known as STFT multipliers or time-frequency localization operators [8, 5, 3, 6].

Using the approximation of the projector from the previous section, we construct an approximation of the multiplier M_m . For a relatively compact neighborhood of the identity U , let $M_{m,U} : \mathcal{S} \rightarrow \mathcal{S}$ be defined by

$$M_{m,U}(f) := P P_U(mf) \quad (f \in \mathcal{S}).$$

Now Theorem 1 implies the following.

Theorem 2. *For each $m \in L^\infty(\mathcal{G})$, $M_{m,U} \rightarrow M_m$ in operator norm, as U ranges over the class of relatively compact neighborhoods of the identity, ordered by inclusion. Moreover, convergence is uniform on any bounded class of symbols.*

Proof. By Proposition 3, for $f \in \mathcal{S}$,

$$\|M_{m,U}(f) - M_m(f)\|_{\mathcal{E}} = \|P P_U(mf) - P P(mf)\|_{\mathcal{E}} \lesssim \|P_U(mf) - P(mf)\|_{\mathcal{E}}.$$

By Theorem 1, $\|P_U(mf) - P(mf)\|_{\mathcal{E}} \lesssim \delta(U)\|mf\|_{W(L^\infty, \mathcal{E})}$, for some function δ such that $\delta(U) \rightarrow 0$, as U grows to \mathcal{G} . Finally, since $m \in L^\infty(\mathcal{G})$ and $f \in \mathcal{S}$, the embedding $\mathcal{S} \hookrightarrow W(L^\infty, \mathcal{E})$ in Proposition 3 implies that $\|mf\|_{W(L^\infty, \mathcal{E})} \lesssim \|f\|_{W(L^\infty, \mathcal{E})} \lesssim \|f\|_{\mathcal{E}}$, and the conclusion follows. Observe that if m belongs to a certain bounded subset of L^∞ , then the last estimate holds uniformly on that set. \square

6. Characterization of the atomic space with multipliers

We can finally prove the main abstract result on the characterization of the atomic space with phase-space multipliers.

Theorem 3. *Under Assumptions (A1), (A2) and (B1), the map*

$$\begin{aligned} C^B : \mathcal{S} &\rightarrow \mathcal{E}_{d,B} \\ f &\mapsto (P(f\eta_\gamma))_\gamma \end{aligned}$$

is left-invertible. Consequently, the following norm equivalence holds for $f \in \mathcal{S}$,

$$\|f\|_{\mathcal{E}} \approx \|(\|P(f\eta_\gamma)\|_B)_\gamma\|_{\mathcal{E}_d}.$$

Remark 8. The fact that there is such a liberty to choose the BF space \mathbf{B} is analogous to the fact that for coorbit spaces only the “global behavior” of the norm imposed on the wavelet transform matters. See [16, Theorem 8.3].

Remark 9. The norm equivalence holds uniformly for any class of spaces \mathbf{E} having the same weight w and the same constant $C_{\mathbf{E},w}$ (cf. Eq. (5)).

Proof of Theorem 3. With the notation of Section 5, using Theorem 2 with symbol $m \equiv 1$, we choose a relatively compact neighborhood of the identity U such that $M_{1,U}$ is invertible. Since the operator P_U (cf. Eq. (13)) can be factored as $P_U = S_U^{\mathbf{B}} C^{\mathbf{B}}$, we have that, $M_{1,U} = P S_U^{\mathbf{B}} C^{\mathbf{B}}$. Since $M_{1,U}$ is invertible, $C^{\mathbf{B}}$ is left-invertible, as claimed. This implies that $\|f\|_{\mathbf{E}} \lesssim \|C^{\mathbf{B}}(f)\|_{E_{d,\mathbf{B}}}$, for $f \in \mathbf{S}$. The converse inequality is just the boundedness of $C^{\mathbf{B}}$ and was proved in Proposition 4. \square

7. The case of atomic decompositions

We now consider a setting where the atomic space from Section 3 has a distinguished atomic decomposition. We prove a number of technical results that will allow us to finely adjust the general results of Section 6 in order to get sharper statements for certain applications.

It is known that under very general conditions any instance of the model introduced in Section 3 has an associated atomic decomposition (see [41]), but nevertheless some matters naturally pertain to the general setting while others are specific to the case of atomic decompositions.

Let us recall Assumption (A1) from Section 3.

- (A1) – \mathbf{E} is a solid, translation invariant BF space, called *the environment*.
 – w is an admissible weight for \mathbf{E} .
 – \mathbf{S} is a closed complemented subspace of \mathbf{E} , called *the atomic subspace*.

We now state Assumption (A2') introducing new ingredients to the model.

- (A2') – $\Lambda \subseteq \mathcal{G}$ is a relatively separated set. Its points will be called *nodes*.
 – $\{\varphi_\lambda \mid \lambda \in \Lambda\}$ and $\{\psi_\lambda \mid \lambda \in \Lambda\}$ are sets of $W^{\text{st}}(L^\infty, L_w^1)$ molecules, enveloped by a function h . That is,
 – $|\varphi_\lambda(x)|, |\psi_\lambda(x)| \leq h(\lambda^{-1}x)$ ($x \in \mathcal{G}$, $\lambda \in \Lambda$),
 – $h \in W^{\text{st}}(L^\infty, L_w^1)$.
 The sets $\{\varphi_\lambda\}_\lambda$ and $\{\psi_\lambda\}_\lambda$ will be called *atoms* and *dual atoms* respectively.
 – $\mathbf{S} \subseteq \mathbf{E}$ has the following atomic decomposition.
 (a) For every $c \in E_d(\Lambda)$, the series $\sum_\lambda c_\lambda \varphi_\lambda$ belong to \mathbf{S} .⁶
 (b) For all $f \in \mathbf{S}$, the following expansion holds

$$f = \sum_{\lambda \in \Lambda} \langle f, \psi_\lambda \rangle \varphi_\lambda. \quad (16)$$

Associated with the atoms we consider the *analysis* and *synthesis* operators given by

⁶ The convergence of the series is clarified in Lemma 2.

$$C : E \rightarrow E_d, \quad C(f) := (\langle f, \psi_\lambda \rangle)_\lambda, \quad (17)$$

$$S : E_d \rightarrow E, \quad S(c) := \sum_\lambda c_\lambda \varphi_\lambda. \quad (18)$$

We also consider their *formal adjoints* given by

$$C' : E_d \rightarrow E, \quad C'(c) := \sum_\lambda c_\lambda \psi_\lambda, \quad (19)$$

$$S' : E \rightarrow E_d, \quad S'(f) := (\langle f, \varphi_\lambda \rangle)_\lambda. \quad (20)$$

Remark 10. Under Assumptions (A1) and (A2'), the operators C, S, C', S' are well defined and bounded by Lemma 2 and the fact that $h \in W^{\text{st}}(L^\infty, L^1_w) \subseteq W_R(L^\infty, L^1_w)$.

We also consider the operator $P : E \rightarrow S$ defined by $P := S \circ C$. Hence,

$$P(f) = \sum_{\lambda \in \Lambda} \langle f, \psi_\lambda \rangle \varphi_\lambda. \quad (21)$$

According to (A2'), P is a projector from E onto S .

We will now see that the setting introduced by (A1) and (A2') can be regarded as an instance of the one set by (A1) and (A2). We first introduce the function H required by (A2). Let $H : \mathcal{G} \rightarrow [0, +\infty)$ be defined by

$$H(x) := \sup_{y \in \mathcal{G}} \sum_{\lambda \in \Lambda} h(\lambda^{-1}y) h(\lambda^{-1}yx). \quad (22)$$

The following lemma shows that P and H satisfy the conditions in (A2).

Lemma 6. *Under Assumptions (A1) and (A2') the following statements hold.*

- (a) *The function H (cf. Eq. (22)) belongs both to $W(L^\infty, L^1_w)$ and $W_R(L^\infty, L^1_w)$.*
- (b) *For every $f \in W(L^1, L^\infty_{1/w})$, the function $P(f) = \sum_\lambda \langle f, \psi_\lambda \rangle \varphi_\lambda$ is well defined (with absolute convergence at every point) and satisfies the following pointwise estimate,*

$$|P(f)(x)| \leq \int_{\mathcal{G}} |f(y)| H(y^{-1}x) dy \quad (x \in \mathcal{G}).$$

Moreover, $\|P(f)\|_{L^\infty_{1/w}} \lesssim \|f\|_{W(L^1, L^\infty_{1/w})} \|H\|_{W_R(L^\infty, L^1_w)}$.

Proof. Part (a) follows from a straightforward computation. One can first establish the estimates,

$$H^\#(x) \lesssim \int_{\mathcal{G}} h_\#(t^{-1})(h_\#)^\#(t^{-1}x) dt, \quad \text{and} \quad H_\#(x) \lesssim \int_{\mathcal{G}} (h_\#)^\#(t^{-1})(h_\#)(t^{-1}x) dt,$$

and then deduce that $\|H\|_{W(L^\infty, L^1_w)} + \|H\|_{W_R(L^\infty, L^1_w)} \lesssim \|h\|_{W^{\text{st}}(L^\infty, L^1_w)}^2$.

Using the enveloping condition in (A2') we get the desired pointwise estimate for P . Part (b) then follows from part (a) and Lemma 1. \square

7.1. Weak continuity of the atomic decomposition of S

Suppose that Assumptions (A1) and (A2') hold. Lemmas 1 and 2 give the embeddings $E \hookrightarrow W(L^1, L_{1/w}^\infty)$ and $E_d \hookrightarrow \ell_{1/w}^\infty$. We denote by $(E_d, \ell_{1/w}^1)$ the space E_d considered with the restriction of the weak* star topology of $\ell_{1/w}^\infty$. Likewise, since by Lemma 1, $W(L^1, L_{1/w}^\infty)$ embeds into the dual space of $W(L^\infty, L_w^1)$, we let $(E, W(L^\infty, L_w^1))$ stand for space E considered with the topology induced by the linear functionals obtained by integration against $W(L^\infty, L_w^1)$ functions. Observe that, since this family of functionals separates points, the corresponding topology is Hausdorff.

We will now establish the continuity of the maps that implement the atomic decomposition of S with respect to these coarser topologies. This will allow us to use density arguments for S . This is irrelevant when the atomic decomposition in Eq. (16) converges in the norm of E , but is important to make the abstract results fully applicable.

Proposition 5. *Under Assumptions (A1) and (A2') the following statements hold.*

- (a) *The map $C : (E, W(L^\infty, L_w^1)) \rightarrow (E_d, \ell_{1/w}^1)$ is continuous.*
- (b) *For $c \in E_d$, the series defining $S(c)$ converge unconditionally in the $(E, W(L^\infty, L_w^1))$ topology. Moreover, the map $S : (E_d, \ell_{1/w}^1) \rightarrow (E, W(L^\infty, L_w^1))$ is continuous.*

Similar statements hold for the operators C' and S' (cf. Eqs. (19) and (20)).

Proof. The operators C, S, C', S' are formally related by

$$\begin{aligned}\langle C(f), c \rangle &= \langle f, C'c \rangle, \\ \langle S(c), f \rangle &= \langle c, S'f \rangle,\end{aligned}$$

with $f \in E$ and $c \in E_d$. The proposition follows easily from here. All the technical details on interchange of summation and integration can be justified using Lemmas 1 and 2. \square

8. More general partitions of unity

Under additional assumptions we can extend Theorem 3 to the case where the condition on the partition of unity: $\sum_\gamma \eta_\gamma \equiv 1$ is relaxed to: $0 < A \leq \sum_\gamma \eta_\gamma \leq B < \infty$. To avoid altering the ongoing notation we keep the assumption that $\sum_\gamma \eta_\gamma \equiv 1$ and introduce a new (generalized) partition of unity $\{\theta_\gamma : \gamma \in \Gamma\}$ related to the one in Assumption (B1) by $\theta_\gamma = m\eta_\gamma$, where $0 < A \leq m \leq B < \infty$. This is the general form of a family of functions $\{\theta_\gamma : \gamma \in \Gamma\}$ enveloped by g and whose sum is non-negative and bounded away from zero and infinity.

Consider the setting of Section 7. The problem of extending Theorem 3 to this new partition of unity can be reduced to the one of establishing the invertibility of the multiplier M_m (cf. Eq. (15)). To this end, we will extend the atomic decomposition on Eq. (16) to an adequate Hilbert space H , then prove the invertibility of M_m on H and finally use the spectral invariance of a certain subalgebra of the algebra of bounded operators on ℓ^2 to deduce the invertibility of

M_m on \mathcal{S} . This is where certain restrictions on the geometry of \mathcal{G} need to be imposed. For the case of time-frequency decompositions and modulation spaces, this line of reasoning is hinted on the final remark of [6] and developed for a very general class of symbols and weighted modulation spaces in [34].

Suppose that Assumptions (A1) and (A2') hold (cf. Section 7). We will now introduce Assumptions (C1) and (C2) and present the extension of Theorem 3.

8.1. Assumption (C1)

We will use a key result from [20]. To this end we introduce the following conditions for a discrete group Ω and a weight u on it.

Definition 1. We say that the pair (Ω, u) satisfies the FGL-conditions if the following hold.

- Ω is a discrete, amenable, rigidly symmetric group.
- $u : \Omega \rightarrow [1, \infty)$ is a submultiplicative, symmetric weight that satisfies $u(e) = 1$ and

$$\lim_{n \rightarrow +\infty} \sup_{x \in U^n} u(x)^{1/n} = 1, \quad \text{and} \\ \inf_{x \in U^n \setminus U^{n-1}} u(x) \approx \sup_{x \in U^n \setminus U^{n-1}} u(x) \quad (n \in \mathbb{N}),$$

for some symmetric generating subset U of Ω , containing the identity element.

For an explanation of the FGL-conditions and their relation to other notions for groups (such as polynomial growth) see [20,19] and the references therein. In Proposition 7 we give more concrete sufficient conditions for the applicability of the FGL-conditions to our setting.

Now we introduce the following assumption on the geometry of \mathcal{G} and the set of nodes Λ that provides the atomic decomposition of \mathcal{S} . This condition will be satisfied in the applications to time-frequency analysis but not in the case of time-scale decompositions.

(C1) We assume the following.

- \mathcal{G} is an IN group.⁷
- The set Λ is a closed, discrete subgroup of \mathcal{G} that, considered as a topological group in itself, satisfies the FGL-conditions with respect to the restriction of the weight w .

Remark 11. The fact that \mathcal{G} is an IN group implies that it is unimodular (i.e. $\Delta \equiv 1$) (see [42]). As a consequence, the weight w is symmetric (i.e. $w(x) = w(x^{-1})$).

The submultiplicativity of w now implies that $(1/w)(xy) \leq w(x)(1/w)(y)$. This equation in turn implies that the weight w is admissible for all the spaces L_w^p and $L_{1/w}^p$ ($1 \leq p \leq +\infty$).

Since under Assumption (C1) Λ is a subgroup, it is possible to consider convolution operators on $E_d(\Lambda)$. The space $E_d(\Lambda)$ is always left invariant, but for a general group \mathcal{G} it may not be

⁷ Remember that, by convention, we also assume that the distinguished neighborhood V is invariant under inner automorphisms.

right invariant (even if E is). Using the fact that in (C1) \mathcal{G} is assumed to be an IN group, the following proposition can be easily proved.

Proposition 6. *Under Assumption (C1), $E_d(\Lambda) * \ell_w^1(\Lambda) \subseteq E_d(\Lambda)$, with the corresponding norm estimate.*

Before introducing the second assumption we give some sufficient conditions for Assumption (C1) to hold. Recall that a group is called almost connected if the quotient by the connected component of the identity element is compact.

Proposition 7. *Suppose that Assumptions (A1), (A2') and (B1) hold and that, in addition, \mathcal{G} is an almost connected IN group. Suppose that Λ is a discrete, closed, finitely-generated subgroup of \mathcal{G} and that the weight w satisfies $w(e) = 1$, the Gelfand–Raikov–Shilov condition,*

$$\lim_{n \rightarrow +\infty} w(\lambda^n)^{1/n} = 1, \quad \text{for all } \lambda \in \Lambda, \quad (23)$$

and the condition,

$$\inf_{x \in U^n \setminus U^{n-1}} w(x) \approx \sup_{x \in U^n \setminus U^{n-1}} w(x), \quad \text{for all } n \in \mathbb{N},$$

for some symmetric generating subset U of Λ , containing the identity.

Then, the conditions in (C1) are satisfied.

Proof. The group \mathcal{G} is an almost connected IN group and therefore has polynomial growth (see [42]). Since Λ is discrete and closed in \mathcal{G} it also has polynomial growth (with respect to the counting measure). Indeed, using the fact that Λ is discrete and closed it follows that there exist W , a compact neighborhood of the identity in \mathcal{G} , such that $\lambda W \cap \lambda' W = \emptyset$ for any two distinct elements $\lambda, \lambda' \in \Lambda$. Then, for any finite set $F \subseteq \Lambda$, the cardinality of F^n is dominated by $|(FW)^n|$.

Hence, Λ is a finitely-generated discrete group of polynomial growth. Therefore Λ is amenable (see [42]). In addition, by Gromov's structure theorem [35], Λ has a nilpotent subgroup of finite index. Corollary 3 from [38] implies that Λ is rigidly symmetric (see also [20]). Finally, since Λ is a finitely-generated discrete group of polynomial growth, Theorem 1.3 from [19] implies that the GRS condition in Eq. (23) implies the condition required in (C1). \square

8.2. Assumption (C2)

In order to introduce the second assumption, suppose that Assumptions (A1), (A2') and (C1) hold and let H be the closed linear subspace of $L^2(\mathcal{G})$ generated by the atoms $\{\varphi_\lambda: \lambda \in \Lambda\}$.

Since \mathcal{G} is now assumed to be unimodular, left and right translations are isometries on $L^2(\mathcal{G})$. Hence, the weight w is also admissible for $L^2(\mathcal{G})$ (cf. Eq. 5) and consequently the operators C and S from Section 7 map $L^2(\mathcal{G})$ into $\ell^2(\Lambda)$ and $\ell^2(\Lambda)$ into $L^2(\mathcal{G})$, respectively (cf. Eqs. (17) and (18)). For clarity, when considered with this domain and codomain we will denote these operators by C_H and S_H . Their adjoints will be denoted by C_H^* and S_H^* . Note that these operators coincide with the maps C' and S' on the intersection of their domains (cf. Eqs. (19) and (20)). We also consider the operator $P_H := S_H C_H$, which coincides with P on $L^2(\mathcal{G}) \cap E$.

Recall that a *frame* for a Hilbert space \mathbf{L} is a collection of vectors $\{e_k\}_k$ such that $\|v\|_{\mathbf{L}} \approx \|(\langle v, e_k \rangle)_k\|_{\ell^2}$, for $v \in \mathbf{L}$. For a general reference on Hilbert-space frames see [51,4]. We now observe that the atoms of \mathbf{S} form a frame for \mathbf{H} .

Claim 1. *The set $\{\varphi_\lambda: \lambda \in \Lambda\}$ is a frame for \mathbf{H} .*

Proof. Since $f = P(f) = P_{\mathbf{H}}(f) = S_{\mathbf{H}}C_{\mathbf{H}}(f)$ for finite linear combinations of the atoms $\{\varphi_\lambda\}_\lambda$, and $C_{\mathbf{H}}$ and $S_{\mathbf{H}}$ are bounded, it follows that $f = S_{\mathbf{H}}C_{\mathbf{H}}(f)$, for all $f \in \mathbf{H}$. This implies that $f = QC_{\mathbf{H}}^*S_{\mathbf{H}}^*(f)$, for all $f \in \mathbf{H}$, where Q is the orthogonal projection onto \mathbf{H} . Hence, $\|f\|_{L^2(\mathcal{G})} \approx \|S_{\mathbf{H}}^*(f)\|_{\ell^2(\Lambda)} = \|(\langle f, \varphi_\lambda \rangle)_\lambda\|_{\ell^2(\Lambda)}$, for all $f \in \mathbf{H}$. \square

Since $\{\varphi_\lambda: \lambda \in \Lambda\}$ is a frame for \mathbf{H} , it has an associated canonical dual frame, that provides an expansion with coefficients having minimal ℓ^2 -norm (see for example [51,4]). This dual frame does not need to coincide with our distinguished set of dual atoms $\{\psi_\lambda: \lambda \in \Lambda\}$. We will now assume that they do coincide. This assumption will be justified in a large number of examples.

(C2) We assume that the set $\{\psi_\lambda: \lambda \in \Lambda\}$ is the canonical dual frame of $\{\varphi_\lambda: \lambda \in \Lambda\}$, considered as a frame for \mathbf{H} .

Under Assumption (C2), $\{\psi_\lambda\}_\lambda \subseteq \mathbf{H}$ and the operator $P_{\mathbf{H}}$ is the orthogonal projector $L^2(\mathcal{G}) \rightarrow \mathbf{H}$. Also, $C_{\mathbf{H}}$ and $S_{\mathbf{H}}$ are related by $C_{\mathbf{H}}^\dagger = S_{\mathbf{H}}$ and $S_{\mathbf{H}}^\dagger = C_{\mathbf{H}}$. (Here L^\dagger denotes the Moore–Penrose pseudo-inverse of an operator L .)

8.3. Convolution-dominated operators

For the remainder of Section 8, we assume that conditions (A1), (A2'), (B1), (C1) and (C2) hold.

Using the fact that Λ is a subgroup, it is possible to dominate operators on \mathbf{E}_d by convolutions. We consider the class of operators dominated by left convolution,

$$CD(\Lambda, w) := \{T \in \mathbb{C}^{\Lambda \times \Lambda} \mid |T_{\lambda, \lambda'}| \leq a_{\lambda\lambda'^{-1}}, \text{ for some } a \in \ell_w^1(\Lambda)\},$$

and we endow it with the norm,

$$\|T\|_{CD(\Lambda, w)} := \inf\{\|a\|_{\ell_w^1} \mid |T_{\lambda, \lambda'}| \leq a_{\lambda\lambda'^{-1}}, \text{ for all } \lambda, \lambda' \in \Lambda\}.$$

$CD(\Lambda, w)$ is a Banach *-algebra (see [20]). We also consider the Banach *-algebra of operators dominated by right convolution,

$$CD_R(\Lambda, w) := \{T \in \mathbb{C}^{\Lambda \times \Lambda} \mid |T_{\lambda, \lambda'}| \leq a_{\lambda'^{-1}\lambda}, \text{ for some } a \in \ell_w^1(\Lambda)\},$$

and we endow it with a norm in a similar manner. We will use a slightly adapted version of the main result from [20].

Proposition 8. *The inclusion $CD_R(\Lambda, w) \hookrightarrow B(\ell^2(\Lambda))$ is spectral (i.e. it preserves the spectrum of each element).⁸ Moreover, if $L \in CD_R(\Lambda, w)$ is a self-adjoint operator with closed range, then its pseudo-inverse L^\dagger also belongs to $CD_R(\Lambda, w)$.*

Proof. Let Λ^{op} denote the set Λ considered with the opposite group operation, given by $\lambda \cdot_{op} \lambda' = \lambda' \lambda$. Since $x \mapsto x^{-1}$ is an algebraic and topological isomorphism between Λ and Λ^{op} and the weight w is symmetric, it follows that Λ^{op} also satisfies the FGL-conditions with respect to the restriction of the weight w . Hence, [20, Corollary 6] implies that $CD(\Lambda^{op}, w)$ is a spectral subalgebra of $B(\ell^2(\Lambda^{op}))$. Finally observe that $CD_R(\Lambda, w) = CD(\Lambda^{op}, w)$ and that $B(\ell^2(\Lambda^{op})) = B(\ell^2(\Lambda))$.

The second part of the theorem is a well-known consequence of the first one. Since the inclusion $CD_R(\Lambda, w) \hookrightarrow B(\ell^2(\Lambda))$ is closed under inversion, it is also closed under holomorphic functional calculus. For a self-adjoint operator with closed range $L \in CD_R(\Lambda, w)$, its pseudo-inverse is given by $L^\dagger = f(L)$, where $f(z) = z^{-1}$, for $z \neq 0$ and $f(0) = 0$. f is holomorphic on the spectrum of L because, since the range of L is closed, 0 is an isolated point of its spectrum. \square

Remark 12. The result in [20] seems to be the most appropriate one for this context but in some cases it is also possible to apply the results in [46,44] to the same end. If the group Λ is \mathbb{Z}^d , then the desired result also follows from [2,32,45] with the advantage of slightly relaxing the assumptions on the weight.

We now observe that $CD_R(\Lambda, w)$ acts on E_d .

Proposition 9. *Let $T \in CD_R(\Lambda, w)$. Then the following hold.*

- (a) T maps E_d into E_d and $\|T\|_{E_d \rightarrow E_d} \lesssim \|T\|_{CD_R(\Lambda, w)}$.
- (b) $T : (E_d, \ell_w^1) \rightarrow (E_d, \ell_w^1)$ is continuous.

Proof. Part (a) follows from Proposition 6 and the solidity of E_d . For part (b), observe that the spaces L_w^1 and $L_{1/w}^\infty$ satisfy the same assumptions that E (cf. Remark 11) and consequently, by part (a), every operator in $CD_R(\Lambda, w)$ maps ℓ_w^1 into ℓ_w^1 and $\ell_{1/w}^\infty$ into $\ell_{1/w}^\infty$. Since the class $CD_R(\Lambda, w)$ is closed under taking adjoints it follows that $T : \ell_{1/w}^\infty \rightarrow \ell_{1/w}^\infty$ is weak* continuous, so part (b) follows. \square

8.4. Invertibility of multipliers

We will now prove the invertibility of M_m on S . We assume that $m \in L^\infty(\mathcal{G})$ is real-valued and satisfies

$$0 < A \leq m \leq B < \infty,$$

for some constants A, B , and we will establish a number of claims that will lead to the desired conclusion.

⁸ Here, $B(\ell^2(\Lambda))$ denotes the algebra of bounded operators on $\ell^2(\Lambda)$.

Claim 2. *The operator $M_m : \mathbf{H} \rightarrow \mathbf{H}$ is invertible.*

Proof. Observe that, since $P_{\mathbf{H}} : L^2(\mathcal{G}) \rightarrow \mathbf{H}$ is the orthogonal projector, and m is real-valued, the operator $M_m : \mathbf{H} \rightarrow \mathbf{H}$ is self-adjoint. Moreover, for $f \in \mathbf{H}$,

$$\begin{aligned} \|M_m(f)\|_{\mathbf{H}} \|f\|_{\mathbf{H}} &\geq \langle P(mf), f \rangle = \langle mf, f \rangle \\ &= \int_{\mathcal{G}} m(x) |f(x)|^2 dx \geq A \|f\|_{\mathbf{H}}^2. \end{aligned}$$

Hence, $M_m : \mathbf{H} \rightarrow \mathbf{H}$ is self-adjoint and bounded below and therefore invertible. \square

Remark 13. Claim 2 may not be true without the assumption that m is non-negative. Indeed, if $\mathcal{G} = \mathbb{R}$, $\Lambda = \mathbb{Z}$, $\varphi_\lambda = \psi_\lambda = \chi_{[\lambda, \lambda+1]}$ and $m = \chi_{(-\infty, 1/2)} - \chi_{[1/2, +\infty)}$, then $M_m(\varphi_0) = 0$.

Let $L \in \mathbb{C}^{\Lambda \times \Lambda}$ be the matrix representing the operator $S_{\mathbf{H}}^* M_m S_{\mathbf{H}} : \ell^2(\Lambda) \rightarrow \ell^2(\Lambda)$. Hence, L is given by

$$L_{\lambda, \lambda'} := \langle m\varphi_{\lambda'}, \varphi_\lambda \rangle.$$

Claim 3. *The matrix L belongs to $CD_R(\Lambda, w)$ and has a Moore–Penrose pseudo-inverse L^\dagger that also belongs to $CD_R(\Lambda, w)$. In addition, $(M_m)^{-1} : \mathbf{H} \rightarrow \mathbf{H}$ can be decomposed as $(M_m)^{-1} = S_{\mathbf{H}} L^\dagger S_{\mathbf{H}}^*$.*

Proof. To see that $L \in CD_R(\Lambda, w)$ let us estimate

$$|L_{\lambda, \lambda'}| \lesssim \int_{\mathcal{G}} h(\lambda^{-1}x) h(\lambda'^{-1}x) dx = a_{\lambda'^{-1}\lambda},$$

where $a_\lambda := h * h^\vee(\lambda)$. Using Lemmas 1 and 2 we see that $a \in \ell_w^1$.

The operator $S_{\mathbf{H}}$ has range \mathbf{H} because $\{\varphi_\lambda\}_\lambda$ is a frame for \mathbf{H} (cf. Claim 1). Since $M_m : \mathbf{H} \rightarrow \mathbf{H}$ is invertible by Claim 2, the range of $L = S_{\mathbf{H}}^* M_m S_{\mathbf{H}}$ equals $S_{\mathbf{H}}^*(\mathbf{H})$. This subspace is closed because $S_{\mathbf{H}}^*$ is bounded below on \mathbf{H} (that is the frame condition). Hence, L has closed range and consequently has a pseudo-inverse L^\dagger . Since M_m is self-adjoint, so is L . In addition, L^\dagger is given by

$$L^\dagger = C_{\mathbf{H}}(M_m)^{-1} C_{\mathbf{H}}^*.$$

Hence, $(M_m)^{-1} = S_{\mathbf{H}} L^\dagger S_{\mathbf{H}}^*$ (where the operator $S_{\mathbf{H}}^*$ is restricted to \mathbf{H}). Finally, by Proposition 8, $L^\dagger \in CD_R(\Lambda, w)$. \square

Now we can prove the invertibility of M_m on S .

Proposition 10. *Let $m \in L^\infty(\mathcal{G})$ be real-valued and satisfy $0 < A \leq m \leq B < \infty$, for some constants A, B . Then, the multiplier $M_m : S \rightarrow S$ is invertible.*

Proof. Let $N_m : S \rightarrow S$ be the operator defined by $N_m := SL^\dagger S'$ (cf. Eq. (20)). It follows from Proposition 9 and Claim 3 that N_m is bounded. Moreover, by Claim 3, for $f \in S \cap H$,

$$M_m N_m(f) = N_m M_m(f) = f. \quad (24)$$

By Propositions 5 and 9, the operators M_m and N_m are continuous in the $(E, W(L^\infty, L_w^1))$ topology. Since by Proposition 5, any $f \in S$ can be approximated by a net of elements of $S \cap H$ in the $(E, W(L^\infty, L_w^1))$ topology (by considering the partial sums of the expansion in Eq. (16)) it follows that Eq. (24) holds for arbitrary $f \in S$. Hence, $M_m : S \rightarrow S$ is invertible. \square

8.5. Characterization of the atomic space with multipliers

Finally we can derive the extension of Theorem 3 to more general partitions of unity.

Theorem 4. Suppose that Assumptions (A1), (A2'), (B1), (C1) and (C2) are satisfied. Let $\{\theta_\gamma : \gamma \in \Gamma\}$ be given by $\theta_\gamma = m\eta_\gamma$, where $0 < A \leq m \leq B < \infty$.

Then the operator,

$$\begin{aligned} \widetilde{C}^B : S &\rightarrow E_{d,B}(\Gamma) \\ f &\mapsto (P(f\theta_\gamma))_\gamma \end{aligned}$$

is left-invertible. Consequently, the following norm equivalence holds for $f \in S$,

$$\|f\|_E \approx \|(\|P(f\theta_\gamma)\|_B)_\gamma\|_{E_d}.$$

Remark 14. Any family $\{\theta_\gamma\}_\gamma$ that is enveloped by g and whose sum is a real-valued function that is bounded away from 0 and ∞ , has the prescribed form for an adequate choice of the partition of unity $\{\eta_\gamma\}_\gamma$ and the function m .

Remark 15. As in Theorem 3, the norm equivalence holds uniformly for any class of spaces E having the same weight w and the same constant $C_{E,w}$ (cf. Eq. (5)).

Proof of Theorem 4. First observe that $\widetilde{C}^B(f) = C^B(mf)$, so \widetilde{C}^B is bounded on S by Propositions 3 and 4. By Proposition 10, M_m is invertible, so by Theorem 2 we can choose a relatively compact neighborhood of the identity U such that $M_{m,U}$ is also invertible. Since the operator P_U (cf. Eq. (13)) can be factored as $P_U = S_U^B C^B$, we have that, $M_{m,U}(f) = P S_U^B C^B(mf) = P S_U^B \widetilde{C}^B(f)$. Since $M_{m,U}$ is invertible, \widetilde{C}^B is left-invertible, as claimed. This also implies the desired norm equivalence. \square

9. Applications

9.1. Coorbit spaces

We now briefly introduce coorbit theory (see [15]) and show how Theorem 3 applies to this context. Let π be a (strongly continuous) unitary representation of \mathcal{G} on a Hilbert space \mathbb{H} . For a fixed $h \in \mathbb{H}$, the abstract wavelet transform is defined as

$$V_h f(x) := \langle f, \pi(x)h \rangle \quad (f \in \mathbb{H}, x \in \mathcal{G}).$$

Let w be an admissible weight on \mathcal{G} . The main assumption in coorbit theory is the existence of a cyclic vector $h \in \mathbb{H}$ that is admissible in the sense that: $V_h h \in W(L^\infty, L_w^1)$ and the reproducing formula,

$$V_h f = V_h f * V_h h,$$

holds for all $f \in \mathbb{H}$. (For a study about the validity of the reproducing formula see [26].) Since $V_h h(x^{-1}) = \overline{V_h h(x)}$, it follows that $V_h h$ also belongs to $W(L^\infty, L_w^1)$. As a consequence of the reproducing formula, $V_h : \mathbb{H} \rightarrow L^2(\mathcal{G})$ is an isometry and therefore has an inverse on its (closed) range.

Under these conditions the space \mathbb{H}_w^1 is defined by

$$\mathbb{H}_w^1 := \{f \in \mathbb{H} \mid V_h f \in L_w^1\},$$

and endowed with the norm $\|f\|_{\mathbb{H}_w^1} := \|V_h f\|_{L_w^1}$. The anti-dual of \mathbb{H}_w^1 (i.e. the space of continuous conjugate-linear functionals) is denoted by $(\mathbb{H}_w^1)^\top$. The inner product $\mathbb{H} \times \mathbb{H} \rightarrow \mathbb{C}$ extends to a sesquilinear form on $\mathbb{H}_w^1 \times (\mathbb{H}_w^1)^\top \rightarrow \mathbb{C}$. Since h is assumed to belong to \mathbb{H}_w^1 , the abstract wavelet transform can be defined for $f \in (\mathbb{H}_w^1)^\top$.

Coorbit spaces are defined by selecting from the reservoir $(\mathbb{H}_w^1)^\top$ those elements that satisfy a certain criteria. Let \mathbf{E} be a solid, translation invariant BF space such that w is admissible for it. The coorbit space is defined by

$$\text{Co } \mathbf{E} := \{f \in (\mathbb{H}_w^1)^\top \mid V_h f \in \mathbf{E}\},$$

and endowed with the norm $\|f\|_{\text{Co } \mathbf{E}} := \|V_h f\|_{\mathbf{E}}$.

Let $\mathbf{S} = V_h(\text{Co } \mathbf{E})$. According to [15, Proposition 4.3], \mathbf{S} is a closed subspace of \mathbf{E} and moreover $P(F) := F * V_h h$ defines a projector onto \mathbf{S} . Hence, if we let $H := |V_h h|$, Assumptions (A1) and (A2) are verified. When \mathbf{E} is $L^2(\mathcal{G})$, the operator P is in fact the orthogonal projector onto \mathbf{S} .

In order to apply Theorem 3 to this setting, let a partition of unity $\{\eta_\gamma\}_\gamma$ and a BF space \mathbf{B} satisfying (B1) be given. Let the operators $M_\gamma : \text{Co } \mathbf{E} \rightarrow \text{Co } \mathbf{E}$ be defined as

$$M_\gamma(f) := V_h^*(\eta_\gamma V_h(f)).$$

Observe that, since $V_h : \mathbb{H} \rightarrow L^2$ is an isometry, V_h^* is the projection onto the range of V_h followed by the inverse of V_h on its range. Hence,

$$V_h M_\gamma(f) := M_{\eta_\gamma} V_h(f),$$

where $M_{\eta_\gamma} : \mathbf{S} \rightarrow \mathbf{S}$ is the multiplier from Section 5. Now Theorem 3 yields the following.

Theorem 5 (Characterization of coorbit spaces). *Let a partition of unity $\{\eta_\gamma\}_\gamma$ and a BF space \mathbf{B} satisfying (B1) be given. Then, for $f \in \text{Co } \mathbf{E}$, the following norm equivalence holds,*

$$\|f\|_{\text{Co } \mathbf{E}} \approx \left\| \left\{ \|M_\gamma(f)\|_{\text{Co } \mathbf{B}} \right\}_\gamma \right\|_{\mathbf{E}_d}.$$

Moreover, the norm equivalence holds uniformly for any class of coorbit spaces $\text{Co } \mathbf{E}$ having the same weight w and the same constant $C_{E,w}$ (cf. Eq. (5)).

In addition, $f \in (\mathbb{H}_w^1)^\perp$ belongs to $\text{Co } \mathbf{E}$ if and only if $\{\|M_\gamma(f)\|_{\text{Co } \mathbf{B}}\}_\gamma \in E_d(\Gamma)$.

Remark 16. One possible choice for \mathbf{B} is $L^2(\mathcal{G})$ yielding $\text{Co } \mathbf{B} = \mathbb{H}$ (cf. [15, Corollary 4.4]).

Proof of Theorem 5. The norm equivalence follows directly from Theorem 3 and the fact that $V_h : \text{Co } \mathbf{E} \rightarrow \mathbf{S}$ is an isometry. The “in addition” part follows from a standard approximation argument. \square

9.2. Time-scale decompositions

We now consider the affine group $\mathcal{G} = \mathbb{R}^d \times (0, +\infty)$, where multiplication is given by $(x, s) \cdot (x', s') = (x + sx', ss')$. Haar’s measure has density $dx \frac{ds}{s^{d+1}}$ and the modular function is given by $\Delta(x, s) = s^{-d}$. The affine group acts on $L^2(\mathbb{R}^d)$ by translations and dilations,

$$\pi(x, s)f(y) = s^{-d/2} f\left(\frac{y-x}{s}\right).$$

The Wavelet transform associated with π is

$$W_h f(x, s) = s^{-d/2} \int_{\mathbb{R}^d} f(t) \overline{h\left(\frac{t-x}{s}\right)} dt,$$

for $f, h \in L^2(\mathbb{R}^d)$, whereas the inverse wavelet transform is given by

$$W_h^* F(x) = \int_0^{+\infty} \int_{\mathbb{R}^d} F(y, s) \overline{h\left(\frac{x-y}{s}\right)} dx \frac{ds}{s^{\frac{3}{2}d+1}},$$

for $F \in L^2(\mathcal{G})$.⁹

The wavelet multiplier with symbol $m \in L^\infty(\mathcal{G})$ is given by

$$\text{WM}_m f(x) = W_h^*(m W_h F), \quad (25)$$

for $f \in L^2(\mathbb{R}^d)$.

The class of coorbit spaces for π contains a large range of the classical function spaces (see [27]) including the Besov and Triebel–Lizorkin spaces. We illustrate Theorem 5 for homogeneous Besov spaces. For $1 \leq p, q \leq +\infty$ and $\sigma \in \mathbb{R}$, the homogeneous Besov space $\dot{B}_{pq}^\sigma(\mathbb{R}^d)$ is the set of all tempered distributions (modulo polynomials) $f \in \mathcal{S}'/\mathcal{P}(\mathbb{R}^d)$ such that

$$\|f\|_{\dot{B}_{pq}^\sigma} := \left(\sum_{j \in \mathbb{Z}} 2^{j\sigma q} \|\mathcal{F}^{-1}(\varphi_j \mathcal{F}(f))\|_{L^p}^q \right)^{1/q}$$

⁹ The integral converges in the weak-sense. The possibility of evaluating it pointwise requires further hypothesis.

is finite (with the usual modification for $q = \infty$), where \mathcal{F} is the Fourier transform and $\{\varphi_j\}_j$ is an adequate Schwartz class partition of unity subordinated to dyadic crowns. It is also usual to present these spaces in terms of integrability of moduli of continuity rather than frequency truncations. See [47] for details.

One of Triebel's characterization of Besov spaces [48] (see also [27]) implies that¹⁰ $\dot{B}_{pq}^\sigma(\mathbb{R}^d) = \text{Co}(L_{\sigma+d/2-d/q}^{p,q}(\mathcal{G}))$, where

$$\|F\|_{L_{\sigma}^{p,q}} = \left(\int_0^{+\infty} \left(\int_{\mathbb{R}^d} |F(x, s)|^p dx \right)^{q/p} s^{-\sigma q} \frac{ds}{s^{d+1}} \right)^{1/q}.$$

As shown in [33, Section 4.2] the admissibility of the window h is implied by the classical “smooth molecule” conditions involving decay of derivatives and vanishing moments (see [23–25]). For example, any radial Schwartz function h with all moments vanishing is adequate.¹¹

In order to illustrate Theorem 5, we consider a covering of $\mathbb{R}^d \times (0, +\infty)$ of the form,

$$U_{k,j} := 2^j((-1, 1)^d + k) \times (2^{j-1}, 2^{j+1}) \quad (k \in \mathbb{Z}^d, j \in \mathbb{Z}), \quad (26)$$

and let $\{\eta_{k,j}\}_{k,j}$ be a (measurable) partition of unity subordinated to it. The discrete norm of a sequence $\{c_{k,j} \mid k \in \mathbb{Z}^d, j \in \mathbb{Z}\}$ associated with the space $L_{\sigma}^{p,q}$ and the covering in Eq. (26) is (see for example [27,49])

$$\|c\|_{(L_{\sigma}^{p,q})_d} \approx \left(\sum_{j \in \mathbb{Z}} 2^{-jq(\sigma+d/q-d/p)} \left(\sum_{k \in \mathbb{Z}^d} |c_{k,j}|^p \right)^{q/p} \right)^{1/q}.$$

We now obtain the following result.

Theorem 6. *The quantity,*

$$\left(\sum_{j \in \mathbb{Z}} 2^{-j\sigma'q} \left(\sum_{k \in \mathbb{Z}^d} \|WM_{\eta_{k,j}} f\|_{L^2}^p \right)^{q/p} \right)^{1/q},$$

where $\sigma' := \sigma + d/2 - d/p$, is an equivalent norm on \dot{B}_{pq}^σ (with the usual modifications when p or q is ∞).

Remark 17. Observe that Theorem 5 also allows for non-compactly supported partitions of unity, as long as its members are enveloped by a well-concentrated function. Also observe that in the norm equivalence above we can measure the norms of $WM_{\eta_{k,j}} f$ in other Besov spaces besides L^2 .

¹⁰ Triebel's result implies that $\|f\|_{\dot{B}_{pq}^\sigma} \approx \|W_h f\|_{L_{\sigma+d/2-d/q}^{p,q}}$ for an adequate window function h . In [15] it is shown that all admissible windows h induce equivalent norms in the coorbit space.

¹¹ To satisfy the general assumptions of Section 3 we can use the weight $w(x, s) := \max\{s^{-\sigma}, \Delta(x, s)^{-1} s^\sigma\} = \max\{s^{-\sigma}, s^{d+\sigma}\}$.

9.3. Time-frequency decompositions

For $f, h \in L^2(\mathbb{R}^d)$, the *Short-Time Fourier Transform* (STFT) (or *windowed Fourier transform*) is defined by

$$\mathcal{V}_h f(x, \varsigma) = \int_{\mathbb{R}^d} f(y) e^{-2\pi i \varsigma y} \overline{h(y-x)} dy.$$

The translation and modulation operators are given by $T_x f(y) := f(y-x)$ and $M_\varsigma f(y) := e^{2\pi i \varsigma y} f(y)$, so that,

$$\mathcal{V}_h f(x, \varsigma) := \langle f, M_\varsigma T_x h \rangle. \quad (27)$$

If h is suitably normalized, $\mathcal{V}_h : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^{2d})$ is an isometry. The adjoint (inverse) STFT is given by

$$\mathcal{V}_h^* F(x) = \int_{\mathbb{R}^{2d}} F(y, \varsigma) M_\varsigma T_y h(x) dy d\varsigma,$$

so the localization operator with symbol $m \in L^\infty(\mathbb{R}^{2d})$ is given by

$$H_m f(x) = \mathcal{V}_h^*(m \mathcal{V}_h f)(x) = \int_{\mathbb{R}^{2d}} m(y, \varsigma) \mathcal{V}_h f(y, \varsigma) M_\varsigma T_y h(x) dy d\varsigma.$$

If h belongs to the Schwartz class, the definition in Eq. (27) extends to tempered distributions. Modulation spaces are then defined by imposing integrability conditions of the STFT. Let $w : \mathbb{R}^{2d} \rightarrow (0, +\infty)$ be a submultiplicative, even weight that satisfies the GRS condition: $\lim_{n \rightarrow \infty} w(nx)^{1/n} = 1$, for all $x \in \mathbb{R}^{2d}$. Let $v : \mathbb{R}^{2d} \rightarrow (0, +\infty)$ be a w -moderated weight; that is: $v(x+y) \lesssim w(x)v(y)$, for all $x, y \in \mathbb{R}^{2d}$. Assume further that v is moderated by a polynomial weight.¹² For $1 \leq p, q \leq +\infty$, the modulation space $M_v^{p,q}$ is defined as

$$M_v^{p,q} := \{f \in \mathcal{S}'(\mathbb{R}^d) \mid \mathcal{V}_h f \in L_v^{p,q}(\mathbb{R}^{2d})\}$$

where

$$\|F\|_{L_v^{p,q}} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |F(x, \varsigma)|^p v(x, \varsigma)^p dx \right)^{q/p} d\varsigma \right)^{1/q},$$

with the usual modifications when p or q is $+\infty$. $M_v^{p,q}$ is of course given the norm $\|f\|_{M_v^{p,q}} = \|\mathcal{V}_h f\|_{L_v^{p,q}}$. For more details on the STFT and modulations spaces see [28].

¹² This assumption is only made in order to define modulation spaces as subsets of the class of tempered distributions. For a general weight, the space $M_v^{p,q}$ has to be constructed as an abstract coorbit space.

After some normalizations and identifications, modulation spaces can be regarded as coorbit spaces of the Schrödinger representation of the Heisenberg group. We chose however to consider them in the context of Section 7. For $h \in M_w^{1,1}$, $1 \leq p, q \leq \infty$, and w, v as above, we let \mathcal{G} be $\mathbb{R}^d \times \mathbb{R}^d$, $\mathbf{E} := L_v^{p,q}(\mathcal{G})$ and $\mathbf{S} := \mathcal{V}_h(M_v^{p,q})$.

For an adequate lattice¹³ $\Lambda \subseteq \mathbb{R}^{2d}$ the system $\{M_\zeta T_x h \mid (x, \zeta) \in \Lambda\}$ gives rise to an atomic decomposition of $M_v^{p,q}$. Moreover, on $M^2 = M_1^{2,2}$ the dual atoms consist of the Hilbert-space dual frame of $\{M_\zeta T_x h \mid (x, \zeta) \in \Lambda\}$ and are of the form $\{M_\zeta T_x \tilde{h} \mid (x, \zeta) \in \Lambda\}$ for some function $\tilde{h} \in M_w^{1,1}$ (see [17,28]). Hence, if we define $\varphi_{(x,\zeta)} := \mathcal{V}_h(M_\zeta T_x h)$ and $\psi_{(x,\zeta)} := \mathcal{V}_h(M_\zeta T_x \tilde{h})$, the atoms $\{\varphi_\lambda \mid \lambda \in \Lambda\}$ and dual atoms $\{\psi_\lambda \mid \lambda \in \Lambda\}$ provide an atomic decomposition for \mathbf{S} .

Since \mathcal{G} is abelian, left and right amalgam spaces coincide. The envelopes for the atoms and dual atoms are the functions $|\mathcal{V}_h h|$ and $|\mathcal{V}_h \tilde{h}|$.¹⁴ These functions indeed envelope the atoms because of the straightforward relation: $|\mathcal{V}_h(M_\zeta T_x f)| = |\mathcal{V}_h f(\cdot - (x, \zeta))|$ (see [28, Eq. 3.14]). The fact that h and \tilde{h} belong to $M_w^{1,1}$ means that $V_h h$ and $V_h \tilde{h}$ belong to L_w^1 , but it is well known that in this case they also belong to $W(L^\infty, L_w^1)$ (see [28, Proposition 12.1.11]). This fact can also be derived from the norm equivalence in Proposition 3.

Let us now consider a family of functions $\{\theta_\gamma \mid \gamma \in \Gamma\}$ that satisfy

$$0 < A \leq \sum_{\gamma} \theta_\gamma \leq B < \infty.$$

Let us also assume that Γ is a relatively separated subset of \mathbb{R}^{2d} and that there exists a function $g \in L_w^1(\mathbb{R}^{2d})$ such that $|\theta_\gamma(x)| \leq g(x - \gamma)$, for all $x \in \mathbb{R}^{2d}$ and $\gamma \in \Gamma$. We will let the space \mathbf{B} that measures the localized pieces be an unweighted Lebesgue space $L^{r,s}$. We are then in the situation of Section 8 (remember that, since \mathcal{G} is abelian, $L_w^1 = W_R^{\text{weak}}(L^\infty, L_w^1)$ – cf. Proposition 1).

To illustrate Theorem 4 more clearly we further assume that $\Gamma = \Gamma_1 \times \Gamma_2$ for two relatively separated sets $\Gamma_1, \Gamma_2 \subseteq \mathbb{R}^d$. Then we get the following.

Theorem 7. *For all $1 \leq s, t \leq \infty$, the quantity,*

$$\left(\sum_{\gamma_2 \in \Gamma_2} \left(\sum_{\gamma_1 \in \Gamma_1} \|H_{\theta_{(\gamma_1, \gamma_2)}} f\|_{M^{s,t}}^p v(\gamma_1, \gamma_2)^p \right)^{q/p} \right)^{1/q},$$

is an equivalent norm on $M_v^{p,q}$ (with the usual modifications when p or q is ∞).

This generalizes the main result in [10] in two directions. The results in [10] apply only to partitions of unity produced by lattice translations of a single function, whereas Theorem 7 allows for irregular partitions. Secondly, in [10] the space measuring the localized pieces is restricted to be L^2 . In contrast, in Theorem 7 it is possible to measure the localized pieces using the whole range of unweighted modulation spaces.

¹³ By a lattice, we mean a full-rank lattice; i.e., a set of the form $\Lambda = A\mathbb{Z}^{2d}$, where A is an invertible matrix.

¹⁴ For simplicity Assumption (A2') requires the same envelope for both the atoms and the dual atoms, but clearly if they have different envelopes then their sum serves as a common envelope.

The proof in [10] resorts to techniques from rotation algebras and spectral theory to construct an atomic decomposition that is simultaneously adapted to all the localization operators $\{H_{\theta_\gamma} \mid \gamma \in \Gamma\}$. Part of our motivation came from the observation that such an atomic decomposition could be obtained in a more constructive manner by using the technique of *frame surgery*, recently introduced in [43].

9.4. Shearlet spaces

Theorem 3 can also be applied to the recently introduced shearlet coorbit spaces [7]. For $a \in \mathbb{R}^* := \mathbb{R} \setminus \{0\}$ and $s \in \mathbb{R}$, the *parabolic scaling* A_a and the *shear* S_s are defined by

$$A_a := \begin{bmatrix} a & 0 \\ 0 & \operatorname{sgn}(a)\sqrt{|a|} \end{bmatrix}, \quad S_s := \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}.$$

The shearlet group is the set $\mathcal{G} := \mathbb{R}^* \times \mathbb{R} \times \mathbb{R}^2$, together with the operation,

$$(a, s, t) \cdot (a', s', t') := (aa', s + \sqrt{|a|}s', t + S_s A_a t').$$

The shearlet group acts on $L^2(\mathbb{R}^2)$ by

$$\pi(a, s, t)f(x) := |a|^{-3/4} f(A_a^{-1} S_s^{-1}(x - t)).$$

In [7] it is proved that any Schwartz function h with Fourier transform supported on a compact subset of $\mathbb{R}^* \times \mathbb{R}$ is an admissible window. The corresponding wavelet transform,

$$SH_h f(a, s, t) := \langle f, \pi(a, s, t)h \rangle,$$

is called the *continuous shearlet transform*. Using Theorem 3, shearlet coorbit spaces can be described in terms of multipliers of the continuous shearlet transform. See [7] for the relevant explicit calculations on the shearlet group (e.g. description of relatively separated sets).

9.5. Localized frames

Let us briefly point out that Theorem 4 also applies to coorbit spaces of localized frames (see [29,21,1]). If \mathbf{H} is a Hilbert space and $\mathcal{F} = \{f_k\}_{k \in \mathbb{Z}^d}$ is a frame for it, every element in $f \in \mathbf{H}$ has an expansion $f = \sum_k \langle f, f_k \rangle g_k$, where $\{g_k\}_{k \in \mathbb{Z}^d}$ is the so-called canonical dual frame. \mathcal{F} is said to be *localized* if $|\langle f_k, f_j \rangle| \leq a_{k-j}$, for some sequence $a \in l_w^1(\mathbb{Z}^d)$ and a weight like the one considered in Section 8. Frame multipliers are defined by applying a mask to the frame expansion:

$$M_m(f) := \sum_k m_k \langle f, f_k \rangle g_k,$$

where $m \in l^\infty(\mathbb{Z}^d)$. Coorbit spaces $H_v^p(\mathcal{F})$ are defined by imposing ℓ_v^p summability conditions to the coefficients $\langle f, f_k \rangle$ (see [21] for the details). Theorem 4 can be applied using $\mathcal{G} = \Lambda = \mathbb{Z}^d$ and yields a characterization of the spaces $H_v^p(\mathcal{F})$ in terms of frame multipliers. Hence, for example, modulation spaces (cf. Section 9.3) can also be characterized in terms of the so-called Gabor multipliers [18].

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